

Discussion Problems: “ILP Solvers”  
Declarative Methods (JHU 600.432/632)  
Prof. Jason Eisner

1. (a) [2 points] If you were to translate the following LP problem into canonical form, how many variables and constraints would it have? (Do not count the “ $\geq 0$ ” constraints.)

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var p,q,r >= 0;
maximize wealth: p+q+r;
subto c1: p <= 10;
subto c2: 3*p >= 4*q;
subto c3: p - q/2 == r;
```

- (b) [4 points] Write the linear programming dual (LP dual) of the above problem.

- (c) [3 points] In the optimal solution to the dual problem, the dual variable associated with constraint c2 is positive. What do you conclude?

What would you conclude if the dual variable were 0?

2. Consider this linear program  $P$ :

$$\begin{aligned} \max \quad & 100x - 400y + 500z \\ 7x + 5y \leq & 5 \\ x + 2y = & 1 \\ y + z \geq & 5 \\ 2x + z \leq & 7 \\ x, y, z \geq & 0 \end{aligned}$$

[oops, this is not actually feasible, but let’s pretend it is]

- (a) [4 points] When you convert this to standard form ( $A\vec{x} = \vec{b}$ , etc.), how many variables and how many constraints will it have? (Not counting the  $x, y, z \geq 0$  constraints.)

- (b) [4 points] How many of these variables will be  $> 0$  in the optimal solution found by the simplex algorithm?  
(AT LEAST, AT MOST, EXACTLY) \_\_\_\_\_  
(circle one and fill in the blank with a number)
- (c) [3 points] How about a solution that is optimal but *cannot* be found by the simplex algorithm? Does your previous answer still apply? Why or why not?
- (d) [4 points] Our LP problem  $P$  was actually obtained as the LP relaxation of a mixed integer program,  $M$ . That program also constrained  $x, y$  to be integer (though not  $z$ ); we simply dropped that integrality constraint.  
Strengthen  $P$  by finding a new linear constraint that must be true of any solution in which  $x, y$  are integer, and therefore of any solution to  $M$ . (Hint: Chvátal cuts.)
- (e) [6 points] Let  $S$  denote the strengthened version of  $P$  from the previous question.  
What's the relation between the maximum of  $M$ , the maximum of  $P$ , and the maximum of  $S$ ? Indicate the strongest necessary relationships (using  $<, \leq, =, \geq, >$ ).
- (f) [4 points] If you strengthen the LP relaxation  $P$  with enough Chvátal cuts, could that make the relaxed problem become totally unimodular? Explain your reasoning.

- (g) [6 points] One way to solve the  $P$  would be variable elimination (the Fourier-Motzkin method). Rewrite the objective equivalently as

$$\begin{aligned} \max \quad & \text{cost} \\ \text{cost} = & 100x - 400y + 500z \end{aligned}$$

There are now 4 constraints involving  $z$ . If you eliminate  $z$  from the system of constraints, what should replace those 4 constraints?

- (h) [6 points] Write the dual of  $P$  in terms of dual variables  $a, b, \dots$

- (i) [2 points] What can you conclude about the dual optimum if  $x > 0$  at the primal optimum?

- (j) [2 points] What can you conclude about the dual optimum if  $7x + 5y < 5$  at the primal optimum (i.e., the first constraint is loose)?

- (k) [2 points] What can you conclude about the dual optimum if  $7x + 5y = 5$  at the primal optimum (i.e., the first constraint is tight)?

- (l) [4 points] Suppose you are running the simplex algorithm on the dual problem. Partway through, you have found a vertex of the dual polytope that achieves a dual objective value of  $k$ . How does the primal optimum compare to  $k$ ?

3. The JHU Student Activities Commission (SAC) is distributing money among 50 clubs. Club number  $i$  will get  $x_i \geq 0$  dollars. The total social benefit that this allocation brings to JHU is  $\sum_{i=1}^{50} f_i(x_i)$ .

The SAC could simply choose  $\vec{x}$  to maximize that sum, but unfortunately they only have \$1000 to distribute. In addition, they want to appear fair. The variance is defined as  $(\frac{1}{50} \sum_{i=1}^{50} x_i^2) -$

$(\frac{1}{50} \sum_{i=1}^{50} x_i)^2$ . So SAC adheres to a hard constraint that the variance must be  $\leq 100$  (i.e., that the standard deviation must be  $\leq \$10$ ).

**Note:** Variance is a convex function of  $\bar{x}$ .<sup>1</sup>

- (a) [2 points] Clearly the mean  $\frac{1}{50} \sum_{i=1}^{50} x_i$  is \$20 if the SAC distributes all of the money. Yet it is conceivable that distributing all of the money is *not* optimal. If that were true, what surprising conclusion would you draw about the functions  $f_i$ ?
- (b) [3 points] Now assume that the functions  $f_i$  are plausible for this real-world scenario. Would you expect the function  $f_i$  to be CONVEX, CONCAVE, BOTH, or NEITHER? Draw a sketch of a typical  $f_i$  before circling one of the choices above. Then explain in a sentence why you circled the choice you did.
- (c) [3 points] You now have an optimization problem with a quadratic fairness constraint and a linear budget constraint. Use the method of Lagrange multipliers to move these two constraints into the objective. What is the new objective? Are there any new constraints?
- (d) [2 points] Is this new objective likely to be difficult to maximize, in the sense of having local maxima that are not globally optimal? Why or why not?

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<sup>1</sup>This isn't hard to see in the case where you know how much money SAC distributed. If  $\mu = \$20$  is the average grant, then the variance can be expressed as  $\frac{1}{50} \sum_{i=1}^{50} (x_i - \mu)^2$ , which is a sum of convex functions and therefore convex.