Mathematical Programming

especially Integer Linear Programming
and Mixed Integer Programming

Transportation Problem in ECLiPSe

- Vars = [A1, A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4];
- Vars :: 0.0..inf,

Can't recover transportation costs by sending negative amounts

Amount that producer "C" sends to consumer "4"

- A1 + A2 + A3 + A4 $=< 500, % supply constraints
- B1 + B2 + B3 + B4 $=< 300,
- C1 + C2 + C3 + C4 $=< 400,

Production capacity of producer "C"

- A1 + B1 + C1 $= 200, % demand constraints
- A2 + B2 + C2 $= 400,
- A3 + B3 + C3 $= 300,
- A4 + B4 + C4 $= 100,

Total amount that must be sent to consumer "4"

- optimize(min(10*A1 + 8*A2 + 5*A3 + 9*A4 +
  7*B1 + 5*B2 + 5*B3 + 3*B4 +
  11*C1 + 10*C2 + 8*C3 + 7*C4), Cost).

Satisfiable?

Transport cost per unit

example adapted from ECLiPSe website

Mathematical Programming in General

- Here are some variables:
- Vars = [A1, A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4];
- And some hard constraints on them:
- Vars :: 0.0..inf,
- A1 + A2 + A3 + A4 $=< 500,   % supply constraints
- B1 + B2 + B3 + B4 $=< 300,
- C1 + C2 + C3 + C4 $=< 400,
- A1 + B1 + C1 $= 200, % demand constraints
- A2 + B2 + C2 $= 400,
- A3 + B3 + C3 $= 300,
- A4 + B4 + C4 $= 100,
- Find a satisfying assignment that makes this objective function as large or small as possible:
- 10*A1 + 8*A2 + 5*A3 + 9*A4 + 7*B1 + 5*B2 + 5*B3 + 3*B4 + 11*C1 + 10*C2 + 8*C3 + 7*C4

Mathematical Programming in General

- Here are some variables:
- And some hard constraints on them:

what kind of constraints?

- Find a satisfying assignment that makes this objective function as large or small as possible:

what kind of function?

Types of Mathematical Programming

Types of Mathematical Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
integer linear prog. (ILP)	integer	linear inequalities	linear function
mixed integer prog. (MIP)	int&real	linear inequalities	linear function
quadratic programming	real	linear inequalities	quadratic function (hopefully convex)
semidefinite prog.	real	linear inequalities +semidefiniteness	linear function
quadratically constrained programming	real	quadratic inequalities	linear or quadratic function
convex programming	real	convex region	convex function
nonlinear programming	real	any	any

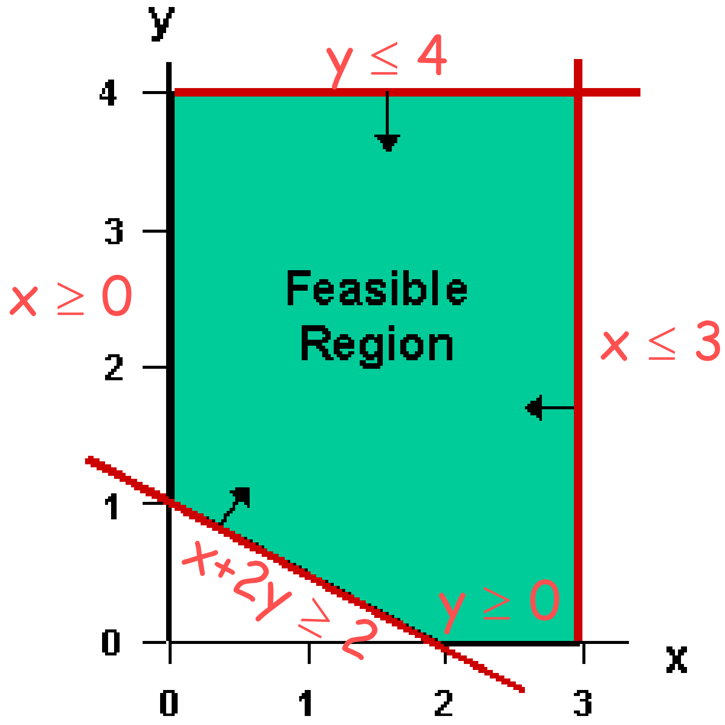
Linear Programming (LP)

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function

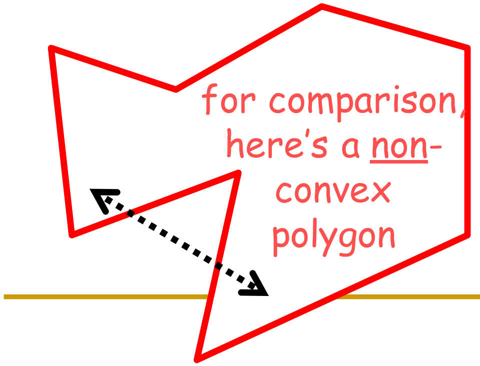
Linear Programming in 2 dimensions

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function

2 variables:
feasible region is a convex polygon



boundary of feasible region comes from the constraints



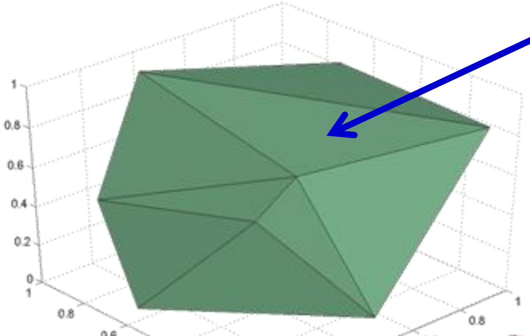
for comparison, here's a non-convex polygon

image adapted from Keely L. Croxton

Linear Programming in *n* dimensions

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function

3 variables:
feasible region is a convex polyhedron



(n-1)-dimensional facet, imposed by a linear constraint that is a full (n-1)-dim hyperplane

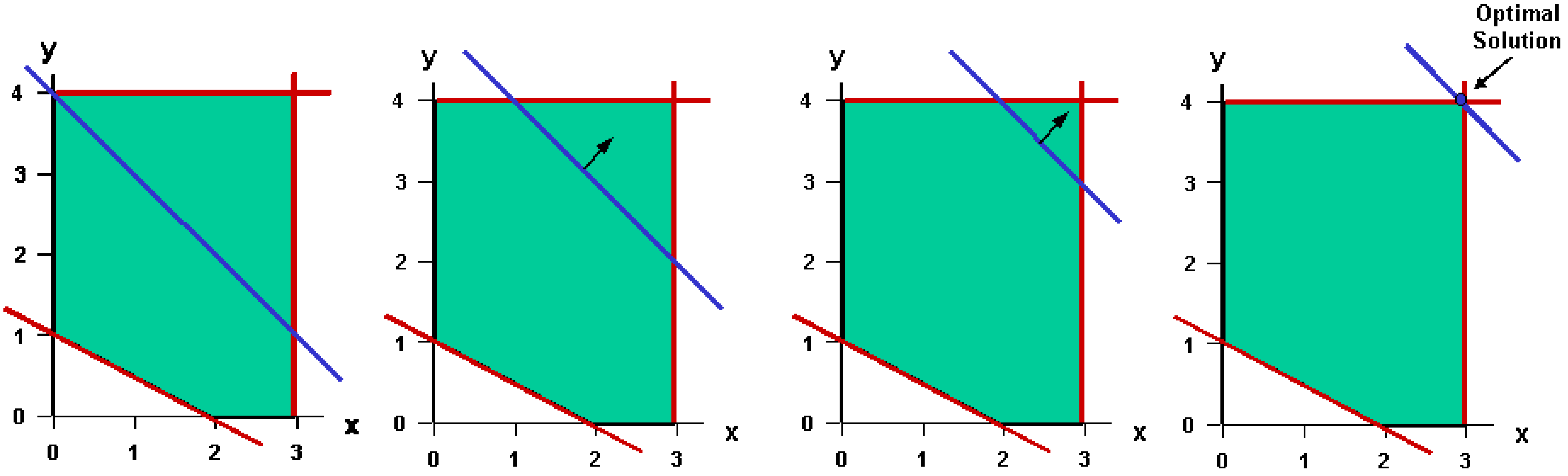
In general case of n dimensions, the word is polytope

image adapted from Keely L. Croxton

Linear Programming in 2 dimensions

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function

"level sets" of the objective x+y (sets where it takes a certain value)



x+y = 4 x+y = 5 x+y = 6 x+y = 7

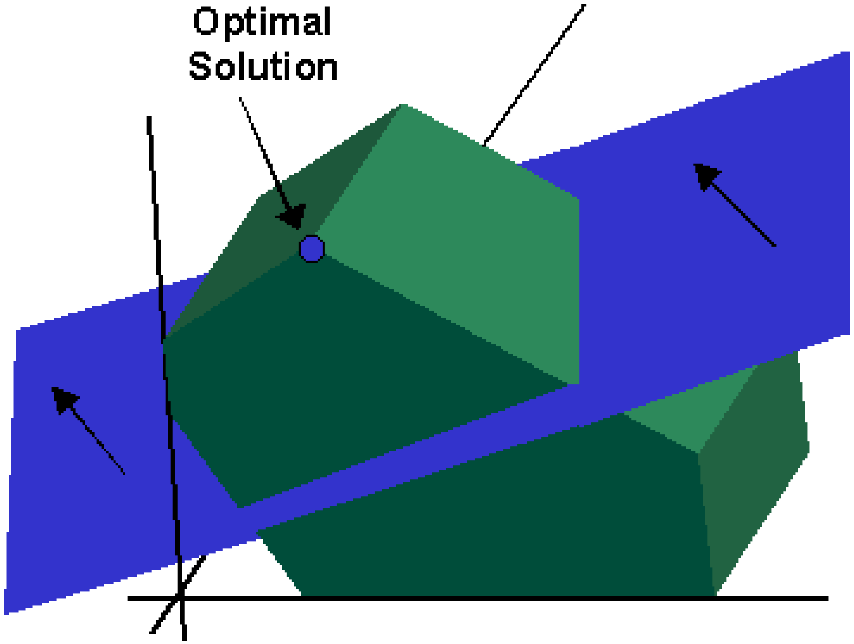
images adapted from Keely L. Croxton

Linear Programming in *n* dimensions

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function

here level set is a plane
(in general, a hyperplane)

If an LP optimum is finite, it can *always* be achieved at a corner ("vertex") of the feasible region.

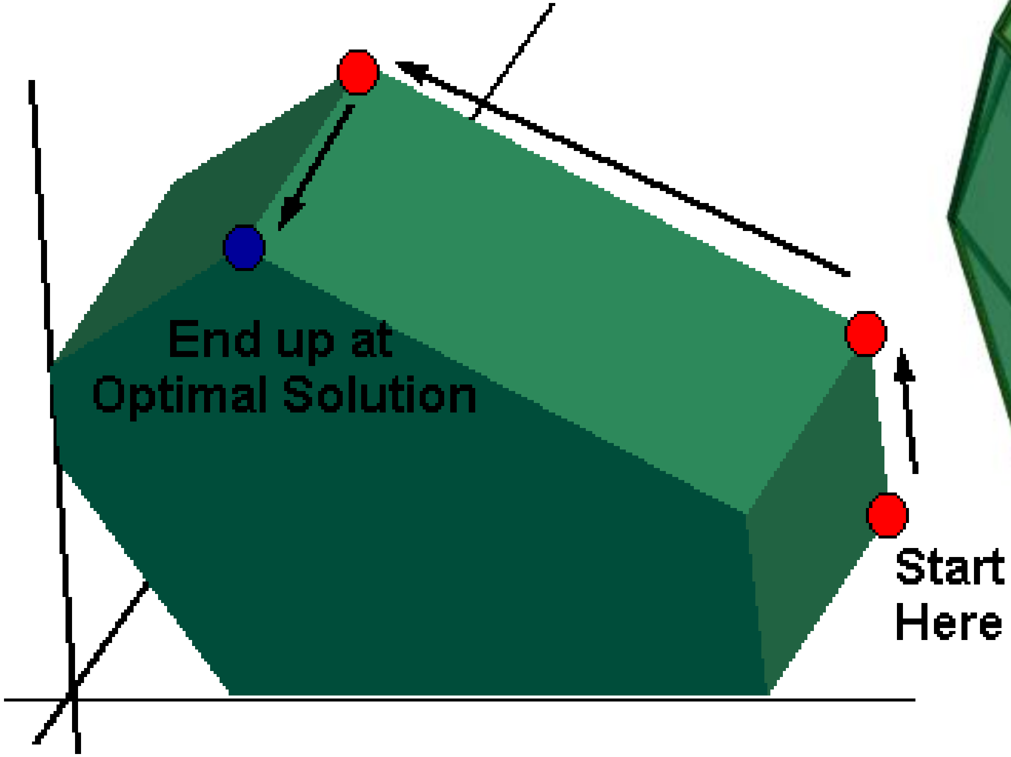


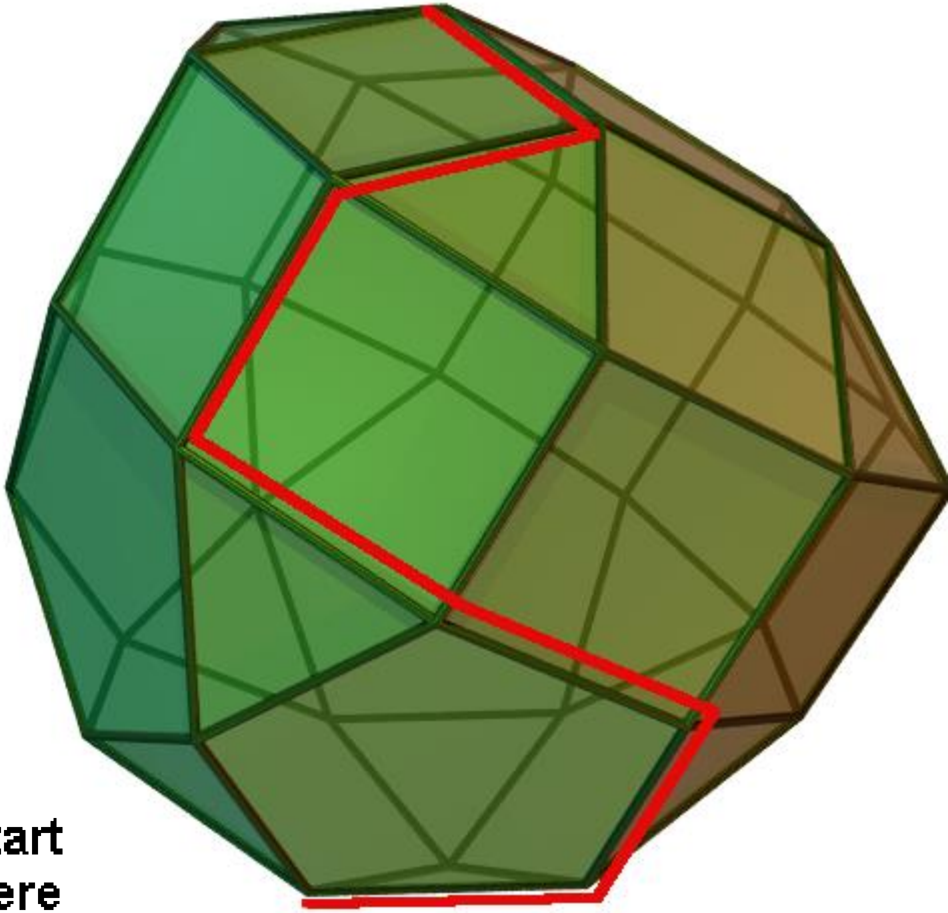
(Can there be infinite solutions? Multiple solutions?)

image from Keely L. Croxton

Simplex Method for Solving an LP

At every step, move to an adjacent vertex that improves the objective.

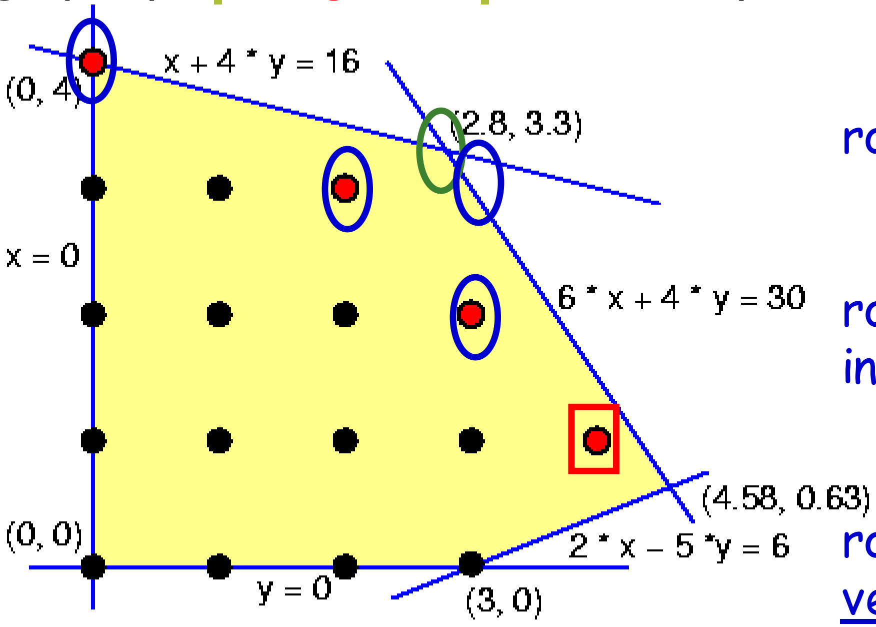




images thanks to Keely L. Croxton and Rex Kincaid

Integer Linear Programming (ILP)

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
integer linear prog. (ILP)	integer	linear inequalities	linear function



round to nearest int (3,3)?
No, infeasible.

round to nearest feasible int (2,3) or (3,2)?
No, suboptimal.

round to nearest integer vertex (0,4)?
No, suboptimal.

Function to maximize: f(x, y) = 6 * x + 5 * y

Optimum LP solution (x, y) = (2.8, 3.3)

Pareto optima: (0, 4), (2, 3), (3, 2), (4, 1)

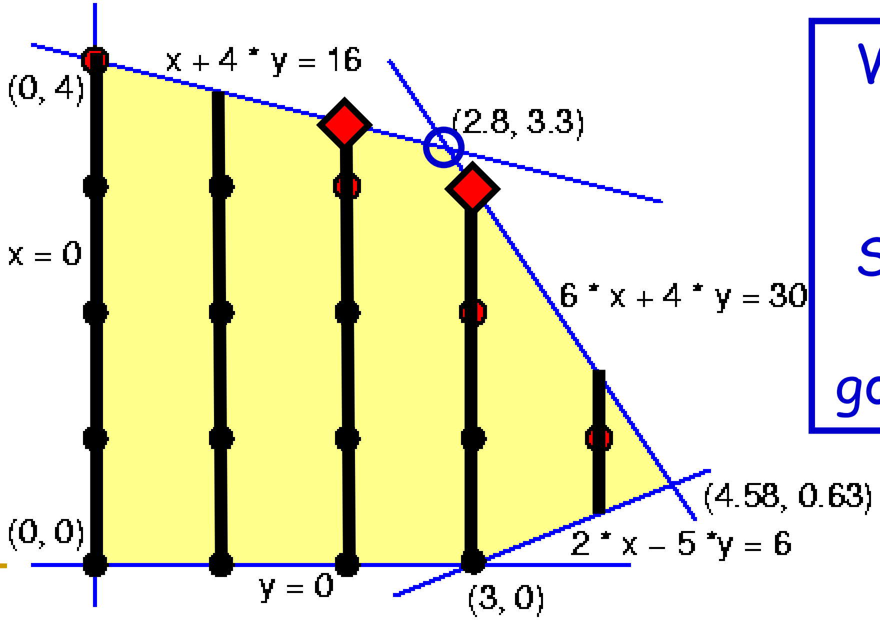
Optimum ILP solution (x, y) = (4, 1)

image adapted from Jop Sibeyn

Mixed Integer Programming (MIP)

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
integer linear prog. (ILP)	integer	linear inequalities	linear function
mixed integer prog. (MIP)	int&real	linear inequalities	linear function

x still integer
but y is now real



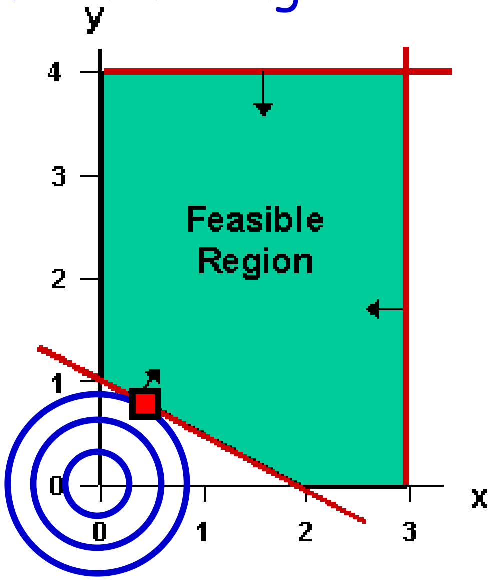
We'll be studying MIP solvers.

SCIP mainly does MIP though it goes a bit farther.

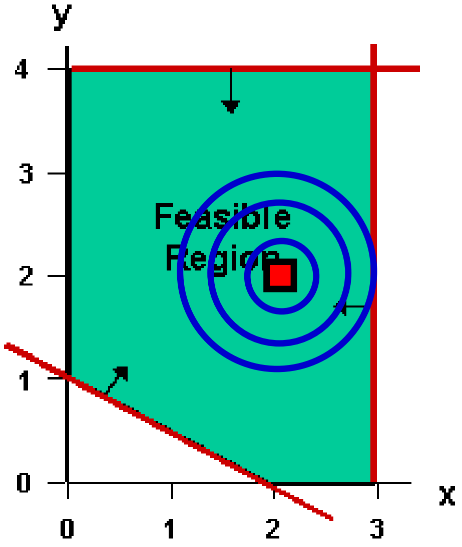
Quadratic Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
quadratic programming	real	linear inequalities	quadratic function (hopefully convex)

solution no longer at a vertex



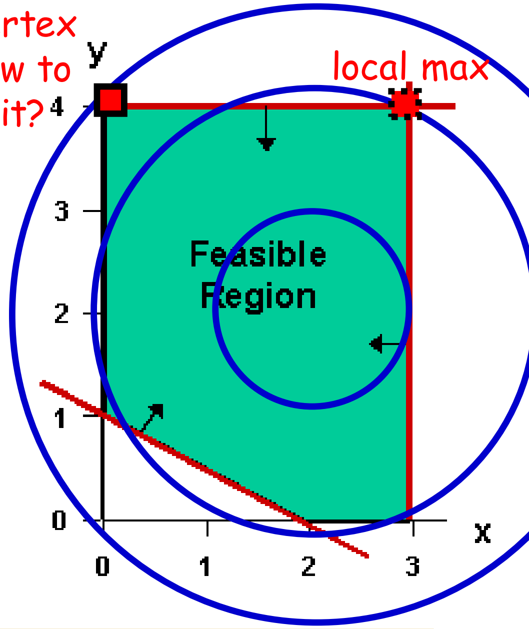
level sets of x^2+y^2
(try to minimize)



level sets of $(x-2)^2+(y-2)^2$
(try to minimize)

at a vertex
but how to
find it?

local max



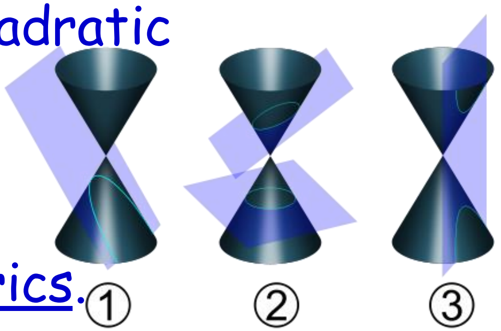
same, but maximize
(no longer convex)

Quadratic Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
quadratic programming	real	linear inequalities	quadratic function (hopefully convex)

Note: On previous slide, we saw that the level sets of our quadratic objective x^2+y^2 were circles.

In general (in 2 dimensions), the level sets of a quadratic function will be conic sections: ellipses, parabolae, hyperbolae. E.g., x^2-y^2 gives a hyperbola.



The n-dimensional generalizations are called quadrics.

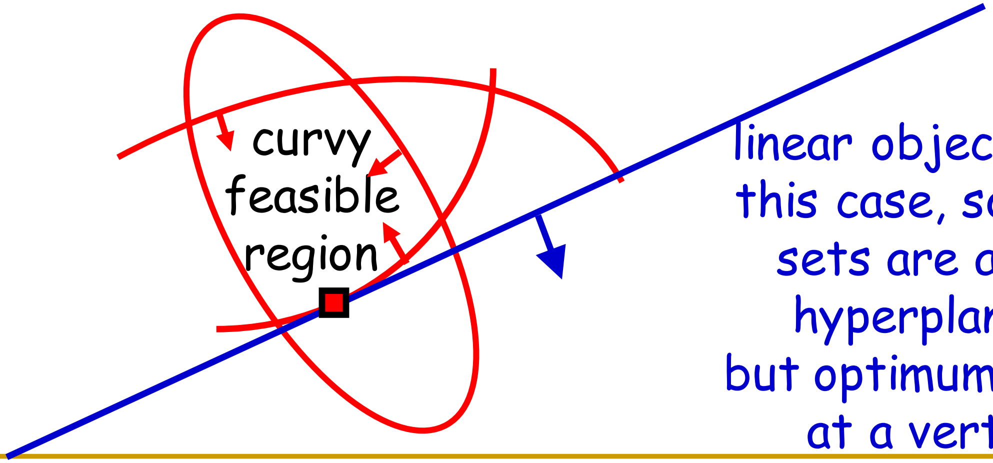
Reason, if you're curious: The level set is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = \text{const}$
Equivalently, $Ax^2 + Bxy + Cy^2 = -Dx - Ey + (\text{const} - F)$
Equivalently, (x,y) is in set if $\exists z$ with $z = Ax^2 + Bxy + Cy^2$ and $z = -Dx - Ey + (\text{const} - F)$
Thus, consider all (x,y,z) points where a right cone intersects a plane

Semidefinite Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
quadratic programming	real	linear inequalities	quadratic function (hopefully convex)
semidefinite prog.	real	linear inequalities +semidefiniteness	linear function

Quadratically Constrained Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
quadratic programming	real	linear inequalities	quadratic function (hopefully convex)
quadratically constrained programming	real	quadratic inequalities	linear or quadratic function



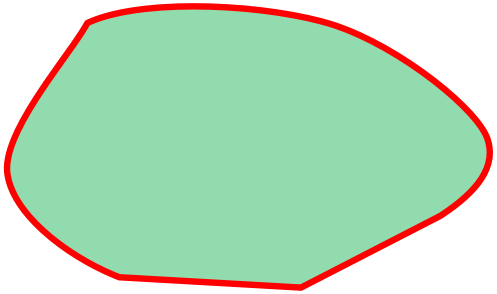
linear objective in this case, so level sets are again hyperplanes, but optimum is not at a vertex

Convex Programming

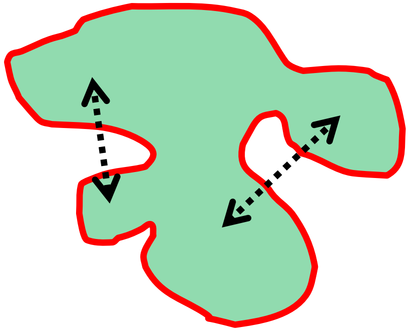
Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
convex programming	real	convex region	convex function (to be minimized)

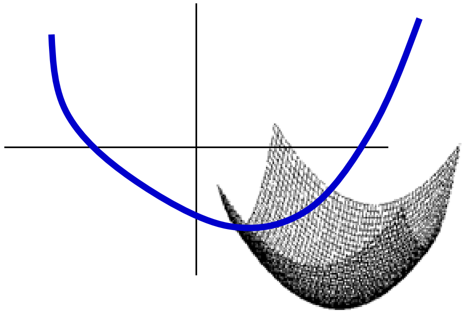
Non-convexity is hard because it leads to disjunctive choices in optimization (hence backtracking search).

- Infeasible in middle of line: which way to go ?
- Objective too large in middle of line: which way to go?

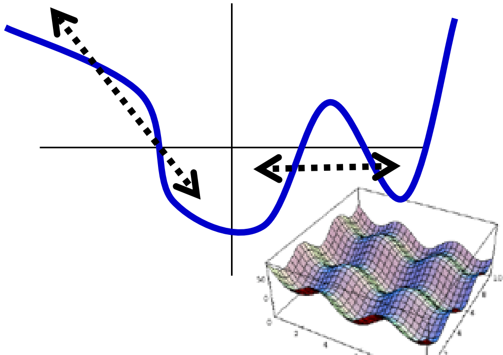


but not





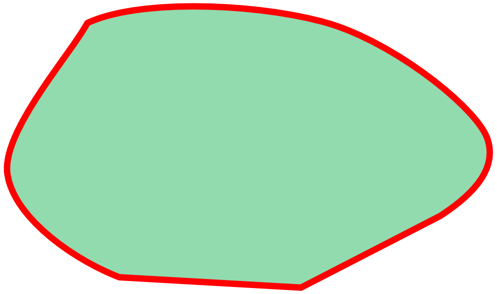
but not

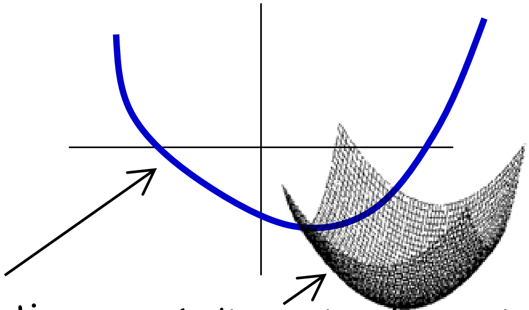


Convex Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
convex programming	real	convex region	convex function (to be minimized)

Can minimize a convex function by methods such as gradient descent, conjugate gradient, or (for non-differentiable functions) Powell's method or subgradient descent.
No local optimum problem.





1st derivative never decreases (formally: 2nd derivative is ≥ 0)

1-dimensional test is met along any line (formally: Hessian is positive semidefinite)

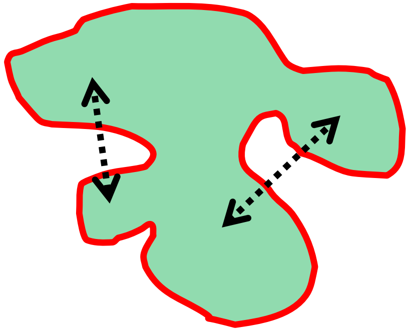
Here we want to generalize to minimization within a convex region. **Still no local optimum problem.** Can use subgradient or interior point methods, etc.

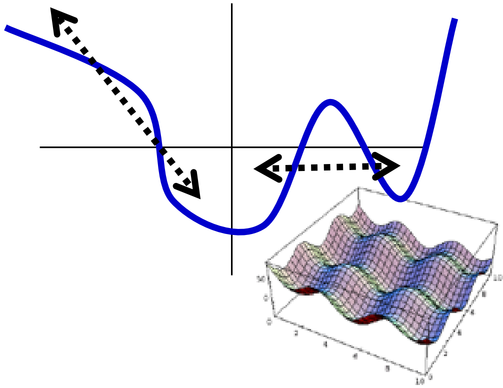
Note: If instead you want to maximize within a convex region, the solution is at least known to be on the boundary, if the region is compact (i.e., bounded).

Nonlinear Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
convex programming	real	convex region	convex function
nonlinear programming	real	any	any

Non-convexity is hard because it leads to disjunctive choices in optimization.





Here in practice one often falls back on methods like simulated annealing.

To get an exact solution, you can try backtracking search methods that recursively divide up the space into regions.

(Branch-and-bound, if you can compute decent optimistic bounds on the best solution within a region, e.g., by linear approximations.)

Types of Mathematical Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear programming (LP)	real	linear inequalities	linear function
integer linear prog. (ILP)	integer	linear inequalities	linear function
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semidefinite prog.	real	linear inequalities +semidefiniteness	linear function
quadratically constrained linear programming	real	quadratic inequalities	linear or quadratic function
convex programming	real	convex region	convex function
nonlinear programming	real	any	any

Types of Mathematical Programming

Name	Vars	Constraints	Objective
constraint programming	discrete?	any	N/A
linear pr			on
integer l			on
mixed ir			on
quadrati			nction onvex)
semidef			on
quadrati linear pr			adratic
convex programming	real	convex region	convex function
nonlinear programming	real	any	any

Lots of software available for
various kinds of math programming!

Huge amounts of effort making it
smart, correct, and fast – use it!

See the NEOS Wiki,
the Decision Tree for Optimization Software,
and the COIN-OR open-source consortium.

Terminology

	Constraint Programming	Math Programming
input	formula / constraint system	model
	variable	variable
	constraint	constraint
	MAX-SAT cost	objective
output	assignment	program
	SAT	feasible
	UNSAT	infeasible
solver	programs	codes
	backtracking search	branching / branch & bound
	variable/value ordering	node selection strategy
	propagation	node preprocessing
	formula simplification	presolving
	{depth,breadth,best,...}-first	branching strategy

Linear Programming in ZIMPL

Formal Notation of Linear Programming

- *n* variables $x_1, x_2, \ldots, x_n$
- max or min objective $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- *m* linear inequality and equality constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

Note: if a constraint refers to only a few of the vars, its other coefficients will be 0

Formal Notation of Linear Programming

- *n* variables $x_1, x_2, \ldots, x_n$
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Note: if a constraint refers to only a few of the vars, its other coefficients will be 0

Formal Notation of Linear Programming

- *n* variables $x_1, x_2, \ldots, x_n$
- max ~~or min~~ objective $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- *m* linear inequality ~~and equality~~ constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad &\leq \quad b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \quad &\leq \quad b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \quad &\leq \quad b_m
\end{aligned}$$

- Can we simplify (much as we simplified SAT to CNF-SAT)?

Formal Notation of Linear Programming

- *n* variables $x_1, x_2, \ldots, x_n$
- objective: max $\vec{c} \cdot \vec{x}$
- *m* linear inequality constraints

$$A\vec{x} \leq \vec{b}$$

(where "$\leq$" means that $(\forall 1 \leq i \leq m)\ (A\vec{x})_i \leq b_i$)

- Now we can use this concise matrix notation

Formal Notation of Linear Programming

- *n* variables $x_1, x_2, \ldots, x_n$
- objective: max $\vec{c} \cdot \vec{x}$
- *m* linear inequality constraints

$$A\vec{x} \leq \vec{b}$$

$$(\text{where ``}\leq\text{'' means that } (\forall 1 \leq i \leq m)\ (A\vec{x})_i \leq b_i)$$

- Some LP folks also assume constraint $\vec{x} \geq 0$
  - What if you want to allow $x_3 < 0$? Just replace x_3 everywhere with $(x_{n+1} - x_{n+2})$ where x_{n+1}, x_{n+2} are new variables ≥ 0.
  - Then solver can pick x_{n+1}, x_{n+2} to have either pos or neg diff.

Strict inequalities?

- *n* variables $x_1, x_2, \ldots, x_n$
- max or min objective $c_1x_1 + c_2x_2 + \ldots + c_nx_n$
- *m* linear inequality and equality constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m
\end{aligned}$$

How about using strict > or < ?
But then you could say "min x_1 subject to $x_1 > 0$."
No well-defined solution, so can't allow this.
Instead, approximate x > y by x ≥ y+0.001.

ZIMPL and SCIP

What little language and solver should we use?

Quite a few options …

- Our little language for this course is ZIMPL (Koch 2004)
  - A free and extended dialect of AMPL = “A Mathematical Programming Language” (Fourer, Gay & Kernighan 1990)
  - Compiles into MPS, an unfriendly punch-card like format accepted by virtually all solvers
- Our solver for mixed-integer programming is SCIP (open source)
  - Our version of SCIP will
    1. read a ZIMPL file (*.zpl)
    2. compile it to MPS
    3. solve using its own MIP methods
       - which in turn call an LP solver as a subroutine
         - our version of SCIP calls CLP (part of the COIN-OR effort)

Transportation Problem in ECLiPSe

- Vars = [A1, A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4];
- Vars :: 0.0..inf,

Can't recover transportation costs by sending negative amounts

Amount that producer "C" sends to consumer "4"

- A1 + A2 + A3 + A4 $=< 500,   % supply constraints
- B1 + B2 + B3 + B4 $=< 300,
- C1 + C2 + C3 + C4 $=< 400,

Production capacity of producer "C"

- A1 + B1 + C1 $= 200, % demand constraints
- A2 + B2 + C2 $= 400,
- A3 + B3 + C3 $= 300,
- A4 + B4 + C4 $= 100,

Total amount that must be sent to consumer "4"

- optimize(min(10*A1 + 8*A2 + 5*A3 + 9*A4 +
  7*B1 + 5*B2 + 5*B3 + 3*B4 +
  11*C1 + 10*C2 + 8*C3 + 7*C4), Cost).

Satisfiable?

Transport cost per unit

example adapted from ECLiPSe website

Transportation Problem in ZIMPL

- var a1; var a2; var a3; var a4;
- var b1; var b2; var b3; var b4;
- var c1; var c2; var c3; var c4;

Amount that producer "C" sends to consumer "4"

Variables are assumed real and >= 0 unless declared otherwise

- subto supply_a: a1 + a2 + a3 + a4 <= 500;
- subto supply_b: b1 + b2 + b3 + b4 <= 300;
- subto supply_c: c1 + c2 + c3 + c4 <= 400;

Production capacity of producer "C"

- subto demand_1: a1 + b1 + c1 == 200;
- subto demand_2: a2 + b2 + c2 == 400;
- subto demand_3: a3 + b3 + c3 == 300;
- subto demand_4: a4 + b4 + c4 == 100;

Total amount that must be sent to consumer "4"

- minimize cost: 10*a1 + 8*a2 + 5*a3 + 9*a4 +
  7*b1 + 5*b2 + 5*b3 + 3*b4 +
  11*c1 + 10*c2 + 8*c3 + 7*c4;

Transport cost per unit

Blue strings are just your names for the constraints and the objective (for documentation and debugging)

Transportation Problem in ZIMPL

- set Producer := {1 .. 3};
- set Consumer := {1 to 4};
- var send[Producer*Consumer];

Indexed variables (indexed by members of a specified set).

Variables are assumed real and >= 0 unless declared otherwise

- subto supply_a: sum <c> in Consumer: send[1,c] <= 500;
- subto supply_b: sum <c> in Consumer: send[2,c] <= 300;
- subto supply_c: sum <c> in Consumer: send[3,c] <= 400;

Indexed summations

- subto demand_1: sum <p> in Producer: send[p,1] == 200;
- subto demand_2: sum <p> in Producer: send[p,2] == 400;
- subto demand_3: sum <p> in Producer: send[p,3] == 300;
- subto demand_4: sum <p> in Producer: send[p,4] == 100;

- minimize cost: 10*send[1,1] + 8*send[1,2] + 5*send[1,3] + 9*send[1,4] +
  7*send[2,1] + 5*send[2,2] + 5*send[2,3] + 3*send[2,4] +
  11*send[3,1] + 10*send[3,2] + 8*send[3,3] + 7*send[3,4];

Transportation Problem in ZIMPL

- set Producer := {“alice”,“bob”,“carol”};
- set Consumer := {1 to 4};
- var send[Producer*Consumer];

(indexed by members of a specified set).

Variables are assumed real and >= 0 unless declared otherwise

- subto supply_a: sum <c> in Consumer: send[“alice”,c] <= 500;
- subto supply_b: sum <c> in Consumer: send[“bob”,c] <= 300;
- subto supply_c: sum <c> in Consumer: send[“carol”,c] <= 400;

- subto demand_1: sum <p> in Producer: send[p,1] == 200;
- subto demand_2: sum <p> in Producer: send[p,2] == 400;
- subto demand_3: sum <p> in Producer: send[p,3] == 300;
- subto demand_4: sum <p> in Producer: send[p,4] == 100;

- minimize cost: 10*send[“alice”,1] + 8*send[“alice”,2] + 5*send[“alice”,3] + 9*senc
  7*send[“bob”,1] + 5*send[“bob”,2] + 5*send[“bob”,3] + 3*send[“b
  11*send[“carol”,1] + 10*send[“carol”,2] + 8*send[“carol”,3] + 7*sen

Transportation Problem in ZIMPL

Variables are assumed real and >= 0 unless declared otherwise

- set Producer := {“alice”,“bob”,“carol”};
- set Consumer := {1 to 4};
- var send[Producer*Consumer] >= -10000;

unknowns (remark: mustn’t multiply unknowns by each other if you want a linear program)

- param supply[Producer] := <"alice"> 500, <"bob"> 300, <"carol"> 400;
- param demand[Consumer] := <1> 200, <2> 400, <3> 300, <4> 100;
- param transport_cost[Producer*Consumer] :=

```
          | 1, 2, 3, 4|
|"alice"|10, 8, 5, 9|
|"bob"  | 7, 5, 5, 3|
|"carol"|11,10, 8, 7|;
```

knowns

- subto supply: forall <p> in Producer:
  (sum <c> in Consumer: send[p,c]) <= supply[p];
- subto demand: forall <c> in Consumer:
  (sum <p> in Producer: send[p,c]) == demand[c];
- minimize cost: sum <p,c> in Producer*Consumer:
  transport_cost[p,c] * send[p,c];

Collapse similar formulas that differ only in constants by using indexed names for the constants, too (“parameters”)

How to Encode Interesting Things in LP (sometimes needs MIP)

Slack variables

- What if transportation problem is UNSAT?
- E.g., total possible supply < total demand

- Relax the constraints. Change

```
subto demand_1: a1 + b1 + c1 == 200;
```

to

```
subto demand_1: a1 + b1 + c1 <= 200 ?
```

No, then we'll manufacture nothing, and achieve a total cost of 0.

Slack variables

- What if transportation problem is UNSAT?
- E.g., total possible supply < total demand

- Relax the constraints. Change

```
subto demand_1: a1 + b1 + c1 == 200;
```

to

```
subto demand_1: a1 + b1 + c1 >= 200 ?
```

Obviously doesn't help UNSAT. But what happens in SAT case?
Answer: It doesn't change the solution. Why not?
Ok, back to our problem …

- This is typical: the solution will achieve equality on some of your inequality constraints. Reaching equality was what stopped the solver from pushing the objective function to an even better value.
- And == is equivalent to >= and <=. Only one of those will be "active" in a given problem, depending on which way the objective is pushing. Here the <= half doesn't matter because the objective is essentially trying to make a1+b1+c1 small anyway. The >= half will achieve equality all by itself.

Slack variables

Also useful if we *could* meet demand but maybe would rather not: trade off transportation cost against cost of not quite meeting demand

- What if transportation problem is UNSAT?
- E.g., total possible supply < total demand

- Relax the constraints. Change

```
subto demand_1: a1 + b1 + c1 == 200;
```

to

```
subto demand_1: a1 + b1 + c1 + slack1 == 200;
```

(or >= 200)

Now add a linear term to the objective:

```
minimize cost: (sum <p,c> in Producer*Consumer:
                    transport_cost[p,c] * send[p,c])
                  + (slack1_cost) * slack1  ;
```

cost per unit of buying from an outside supplier

Slack variables

Also useful if we *could* meet demand but maybe would rather not: trade off transportation cost against cost of not quite meeting demand

- What if transportation problem is UNSAT?
- E.g., total possible supply < total demand

- Relax the constraints. Change

```
subto demand_1: a1 + b1 + c1 == 200;
```

to

```
subto demand_1: a1 + b1 + c1 == 200 - slack1 ;
```

Now add a linear term to the objective:

```
minimize cost: (sum <p,c> in Producer*Consumer:
                    transport_cost[p,c] * send[p,c])
                    + (slack1_cost) * slack1  ;
```

cost per unit of doing without the product

Piecewise linear objective

- What if cost of doing without the product goes up nonlinearly?
- It’s pretty bad to be missing 20 units, but we’d make do.
- But missing 60 units is really horrible (more than 3 times as bad) …

- We can handle it still by linear programming:

```
subto demand_1: a1 + b1 + c1 + slack1 + slack2 + slack3 == 200 ;
subto s1: slack1 <= 20;  # first 20 units
subto s2: slack2 <= 10;  # next 10 units (up to 30)
subto s3: slack3 <= 30;  # next 30 units (up to 60)
```

so max total slack is 60; could drop this constraint to allow ∞

Now add a linear term to the objective:

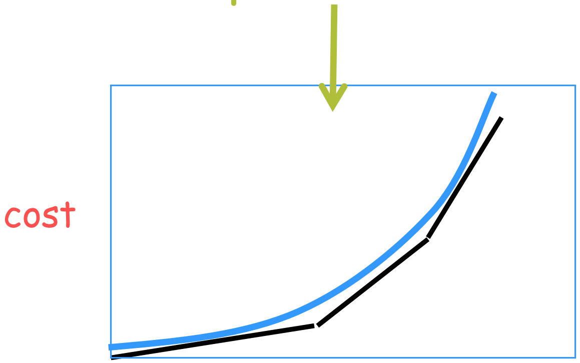
```
minimize cost: (sum <p,c> in Producer*Consumer:
                    transport_cost[p,c] * send[p,c])
+ (slack1_cost * slack1) + (slack2_cost * slack2) + (slack3_cost * sla
```

not too bad   worse (per unit)   ouch! out of business

Piecewise linear objective

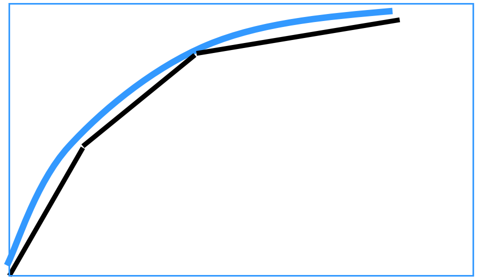
```
subto demand_1: a1 + b1 + c1 + slack1 + slack2 + slack3 <= 200 ;
subto s1: slack1 <= 20;   # first 20 units
subto s2: slack2 <= 10;   # next 10 units (up to 30)
subto s3: slack3 <= 30;   # next 30 units (up to 60)
minimize cost: (sum <p,c> in Producer*Consumer:
                       transport_cost[p,c] * send[p,c])
      + (slack1_cost * slack1) + (slack2_cost * slack2) + (slack3_cost * slack3);
```

Note: Can approximate any continuous function by piecewise linear.
In our problem, slack1 <= slack2 <= slack3 (costs get worse).

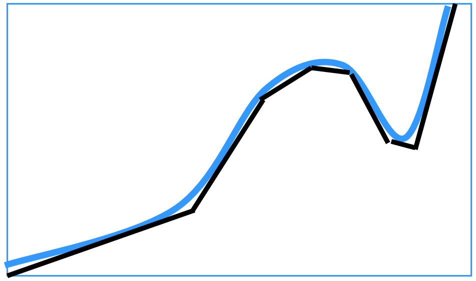


resource being bought
(or amount of slack being suffered)

increasing cost
(diseconomies of scale)
(resource is scarce or critical)



decreasing cost
(economies of scale)
(resource is cheaper in bulk)



arbitrary non-convex function
(hmm, can we optimize this?)

Piecewise linear objective

-
  ```
  subto demand_1: a1 + b1 + c1 + slack1 + slack2 + slack3 <= 200 ;
  subto s1: slack1 <= 20;   # first 20 units
  subto s2: slack2 <= 10;   # next 10 units (up to 30)
  subto s3: slack3 <= 30;  # next 30 units (up to 60)
  minimize cost: (sum <p,c> in Producer*Consumer:
                                   transport_cost[p,c] * send[p,c])
           + (slack1_cost * slack1) + (slack2_cost * slack2) + (slack3_cost * slack3);
  ```

Note: Can approximate any continuous function by piecewise linear.

In our problem, slack1_cost <= slack2_cost <= slack3_cost (costs get worse).

It's actually important that costs get worse. Why?

Answer 1: Otherwise the encoding is wrong!
(If slack2 is cheaper, solver would buy from outside supplier 2 first.)

Answer 2: It ensures that the objective function is convex!
Otherwise too hard for LP; we can't expect any LP encoding to work.

Therefore: E.g., if costs get progressively cheaper, (e.g., so-called "economies of scale" – quantity discounts), then you can't use LP. ☹

How about integer linear programming (ILP)?

Piecewise linear objective

-
  ```
  subto demand_1: a1 + b1 + c1 + slack1 + slack2 + slack3 <= 200 ;
  subto s1: slack1 <= 20;  # first 20 units
  subto s2: slack2 <= 10;  # next 10 units (up to 30)
  subto s3: slack3 <= 30; # next 30 units (up to 60)
  minimize cost: (sum <p,c> in Producer*Consumer:
                          transport_cost[p,c] * send[p,c])
        + (slack1_cost * slack1) + (slack2_cost * slack2) + (slack3_cost * slack3);
  ```

- Need to ensure that *even if the slack_costs are set arbitrarily (any function!),* slack1 must reach 20 before we can get the quantity discount by using slack2.
- Use integer linear programming. How?
- ~~var k1 binary;~~ var k2 binary; var k3 binary; # 0-1 ILP
- subto slack1 <= 20*~~k1~~; # can only use slack1 if k1==1, not if k1==0
  subto slack2 <= 10*k2;
  subto slack3 <= 30*k3;

  If we want to allow ∞ total slack, should we drop this constraint? No, we need it (if k3==0). Just change 30 to a large number M. (If slack3 reaches M in the solution, increase M and try again. ☺)
- subto slack1 >= k2*20; # if we use slack2, then slack1 must be fully used
  subto slack2 >= k3*10; # if we use slack3, then slack2 must be fully used

Can drop k1. It really has no effect, since nothing stops it from being 1. Corresponds to the fact that we're *always* allowed to use slack1.

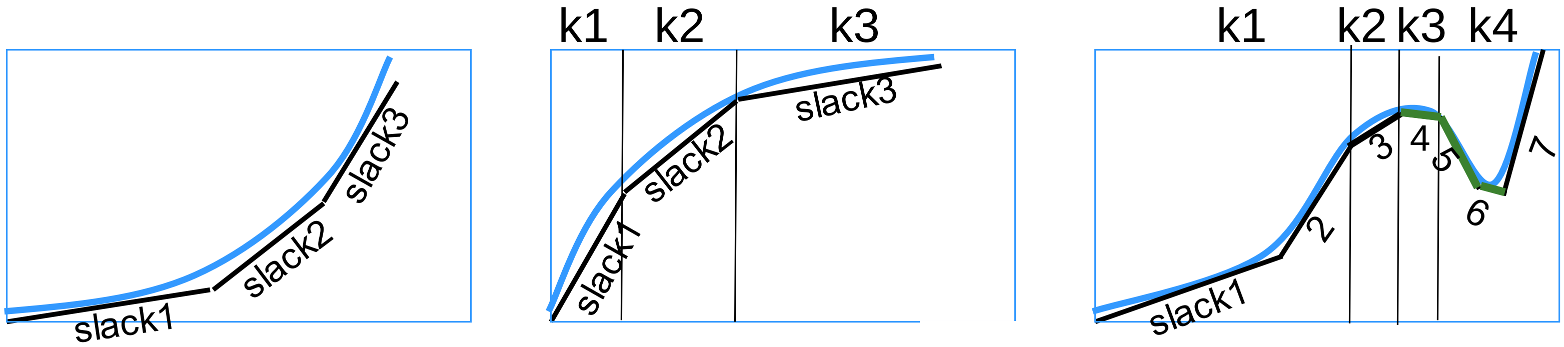
Piecewise linear objective

```
subto demand_1: a1 + b1 + c1 + slack1 + slack2 + slack3 <= 200 ;
subto s1: slack1 <= 20;  # first 20 units
subto s2: slack2 <= 10;  # next 10 units (up to 30)
subto s3: slack3 <= 30; # next 30 units (up to 60)
minimize cost: (sum <p,c> in Producer*Consumer:
                              transport_cost[p,c] * send[p,c])
        + (slack1_cost * slack1) + (slack2_cost * slack2) + (slack3_cost * slack3);
```

Note: Can approximate any continuous function by piecewise linear.

Divide into convex regions, use ILP to choose region.

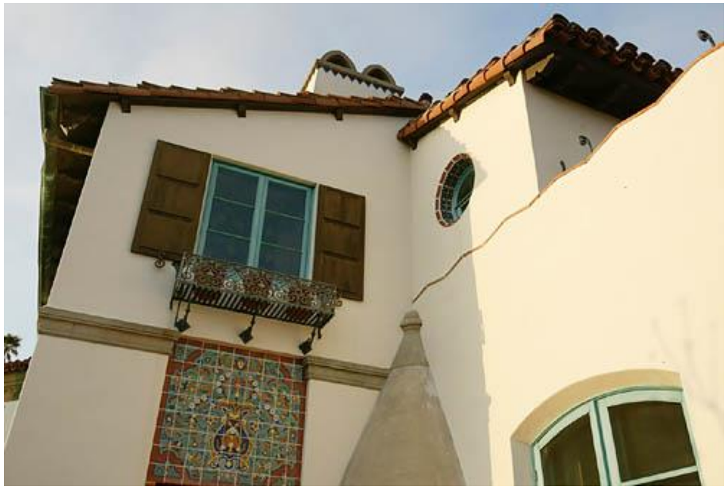
cost



resource being bought
(or amount of slack being suffered)

slack4_cost is negative
slack5_costs is negative
slack6_cost is negative
so in these regions, prefer to take more slack (if constraints allow)

Image Alignment



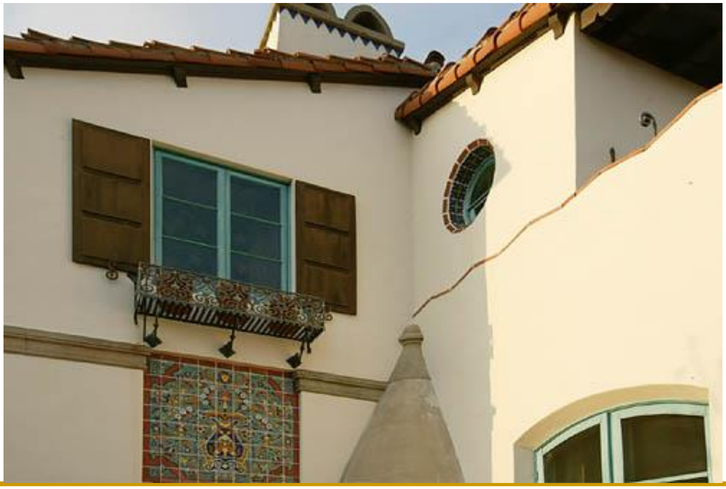
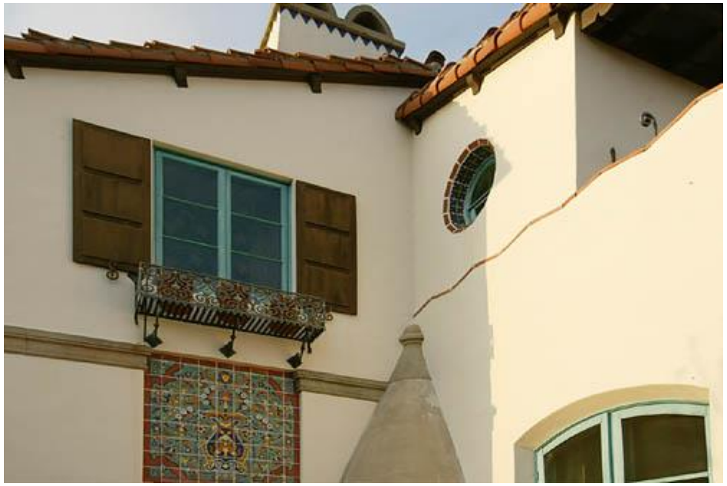


Image Alignment

as a transportation problem, via "Earth Mover's Distance" *(Monge, 1781)*



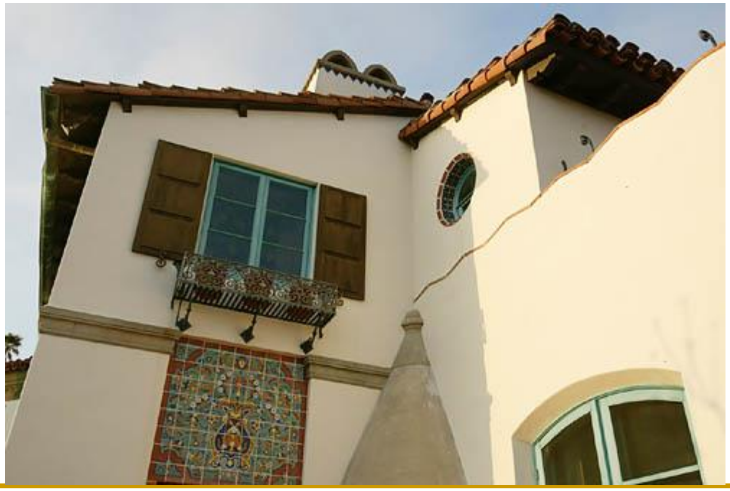
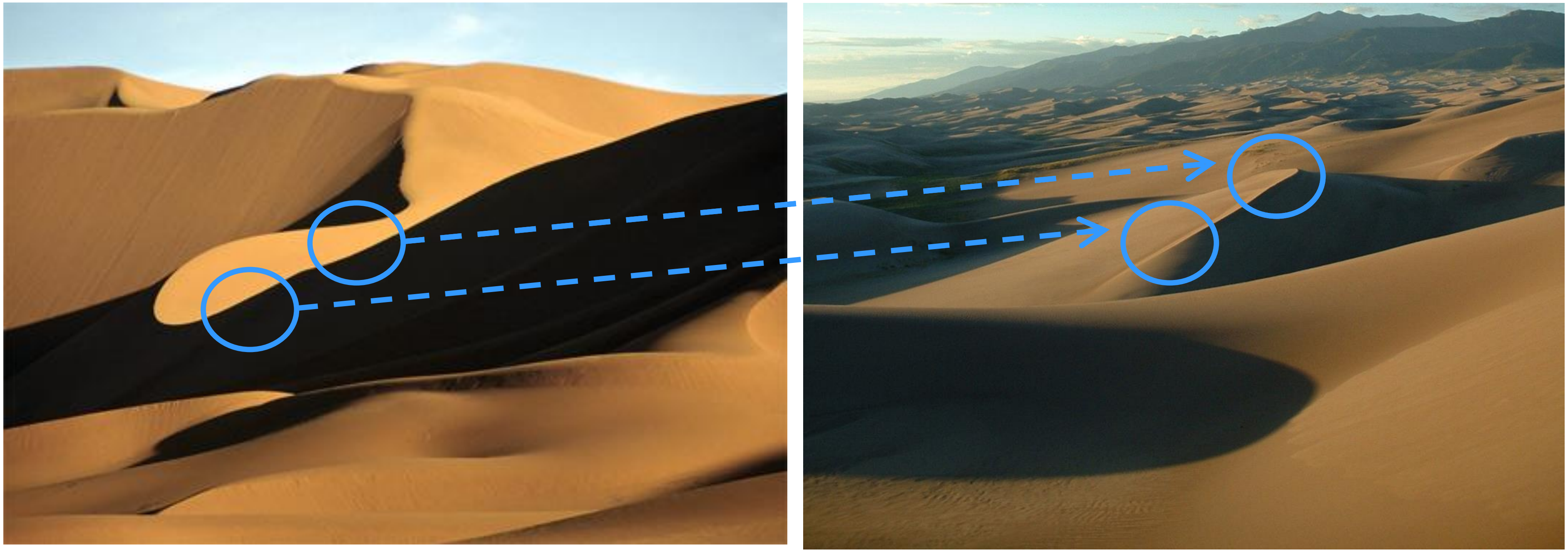


Image Alignment

as a transportation problem, via "Earth Mover's Distance" *(Monge, 1781)*



warning: this code takes some liberties with ZIMPL,
which is not quite this flexible in handling tuples;
a running version would be slightly uglier

Image Alignment

as a transportation problem, via "Earth Mover's Distance" *(Monge, 1781)*

- param N := 12; param M := 10; # dimensions of image
- set X := {0..N-1}; set Y := {0..M-1};
- set P := X*Y; # points in source image
- set Q := X*Y; # points in target image
- defnumb norm(x,y) := sqrt(x*x+y*y);
- defnumb dist(<x1,y1>,<x2,y2>) := norm(x1-x2,y1-y2);
- param movecost := 1;
- param delcost := 1000; param inscost := 1000;
- var move[P*Q]; # amount of earth moved from P to Q
- var del[P]; # amount of earth deleted from P in source image
- var ins[Q]; # amount of earth added at Q in target image

Image Alignment

warning: this code takes some liberties with ZIMPL, which is not quite this flexible in handling tuples; a running version would be slightly uglier

as a transportation problem, via "Earth Mover's Distance" *(Monge, 1781)*

```
defset Neigh := { -1 .. 1 } * { -1 .. 1 } - {<0,0>};
```

```
minimize emd:
   (sum <p,q> in P*Q: move[p,q]*movecost*dist(p,q))
   + (sum <p> in P: del[p]*delcost) + (sum <q> in Q: ins[q]*inscost);
```

```
subto source: forall <p> in P:
       source[p] == del[p] + (sum <q> in Q: move[p,q]);
subto target: forall <q> in Q:
       target[q] == ins[q] + (sum <p> in P: move[p,q]);
```

slack

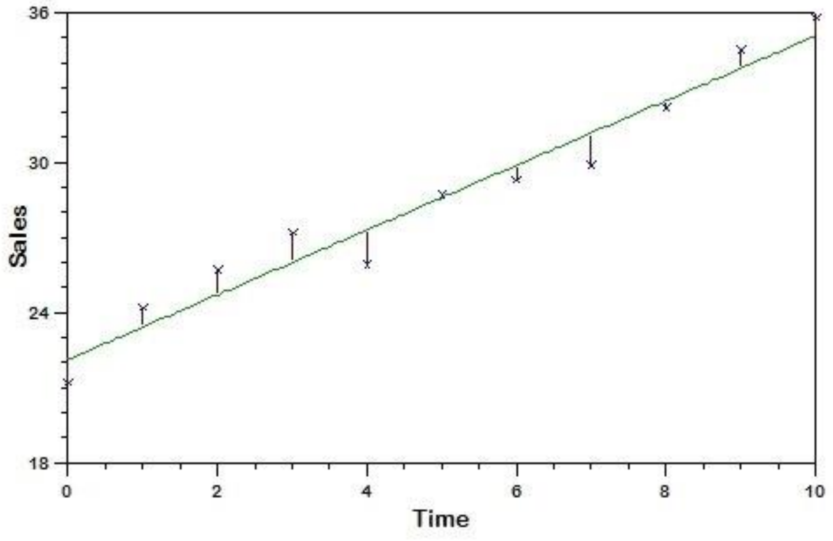
don't have to do it all by moving dirt: if that's impossible or too expensive, can manufacture/destroy dirt)

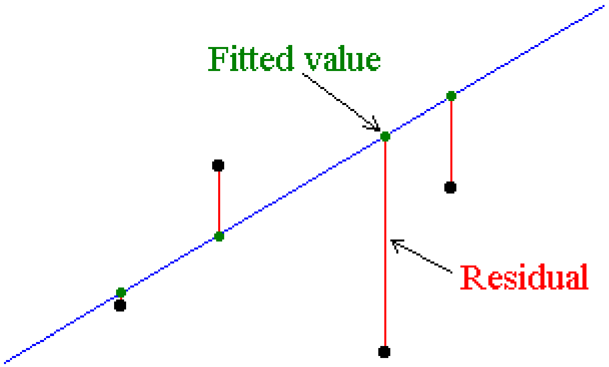
```
subto smoothness: forall <p> in P: forall <q> in Q: forall <d> in Neigh:
       move[p,q]/source[p] <= 1.01*move[p+d,q+d]/source[p+d]
```

constant, so ok for LP (if > 0)

no longer a standard transportation problem; solution might no longer be integers (even if 1.01 is replaced by 2)

L1 Linear Regression





- Given data (x_1,y_1), (x_1,y_2), … (x_n,y_n)
- Find a linear function $y=mx+b$
  that approximately predicts each y_i from its x_i (why?)
- Easy and useful generalization not covered on these slides:
  - each x_i could be a vector (then m is a vector too and mx is a dot product)
  - each y_i could be a vector too (then mx is a matrix and mx is a matrix multiplication)

L1 Linear Regression

- Given data (x_1,y_1), (x_1,y_2), … (x_n,y_n)
- Find a linear function $y=mx+b$
  that approximately predicts each y_i from its x_i
- Standard "L2" regression:
  - minimize $\sum_i (y_i - (mx_i+b))^2$
  - This is a convex quadratic problem. Can be handled by gradient descent, or more simply by setting the gradient to 0 and solving.
- "L1" regression:
  - minimize $\sum_i |y_i - (mx_i+b)|$, so m and b are less distracted by outliers
  - Again convex, but not differentiable, so no gradient!
  - But now it's a linear problem. Handle by linear programming:

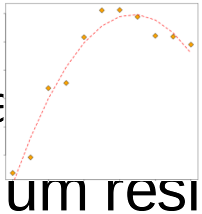
    subto $y_i == (mx_i+b) + (u_i - v_i)$; subto $u_i \geq 0$; subto $v_i \geq 0$;
    minimize $\sum_i (u_i + v_i)$;

More variants on linear regression

- L1 linear regression:
  - minimize $\sum_i |y_i - (mx_i+b)|$, so m and b are less distracted by outliers
  - Handle by linear programming:

    subto $y_i = (mx_i+b) + (u_i - v_i)$;   subto $u_i \geq 0$; subto $v_i \geq 0$;
    minimize $\sum_i (u_i + v_i)$;

- Quadratic regression: $y_i \approx (ax_i^2 + bx_i + c)$?
  - Answer: Still linear constraints! x_i^2 is a consta $(x_{i,}y_i)$ is given.
- L∞ linear regression: Minimize the maximum residual instead of the total of all residuals?
  - Answer: minimize z; subto forall <i> in I: $u_i+v_i \leq z$;
  - Remark: Including max(p,q,r) in the cost function is easy. Just minimize z subject to p ≤ z, q ≤ z, r ≤ z. Keeps all of them small.
  - But: Including min(p,q,r) is hard! Choice about which one to keep small.
    - Need ILP. Binary a,b,c with a+b+c==1. Choice of (1,0,0),(0,1,0),(0,0,1).
    - Now what? First try: min ap+bq+cr. But ap is quadratic, oops!
    - Instead: use lots of slack on unenforced constraints. Min z subj. to p ≤ z+M(1-a), q ≤ z+M(1-b), r ≤ z+M(1-c), where M is large constant.



CNF-SAT (using binary ILP variables)

- We just said “a+b+c==1” for “exactly one” (sort of like XOR).
- Can we do any SAT problem?
  - If so, an ILP solver can handle SAT … and more.
- Example: (A v B v ~C) ^ (D v ~E)
- SAT version:
  - constraints: (a+b+(1-c)) >= 1,   (d+(1-e)) >= 1
  - objective: none needed, except to break ties

- MAX-SAT version:

  slack
  - constraints: (a+b+(1-c))+u1 >= 1,   (d+(1-e))+u2 >= 1
  - objective: minimize c1*u1+c2*u2
    where c1 is the cost of violating constraint 1, etc.

Non-clausal SAT (again using 0-1 ILP)

- If A is a [boolean] variable, then A and ~A are “literal” formulas.
- If F and G are formulas, then so are
  - F ^ G (“F and G”)
  - F v G (“F or G”)
  - F → G (“If F then G”; “F implies G”)
  - F ↔ G (“F if and only if G”; “F is equivalent to G”)
  - F xor G (“F or G but not both”; “F differs from G”)
  - ~F (“not F”)

- If we are given a non-clausal formula, easy to set up as ILP.
  - Use aux variables exactly as in Tseitin transformation.
  - Need only a linear number of new variables and new constraints.

Non-clausal SAT (again using 0-1 ILP)

- If we are given a non-CNF constraint, easy to set up as ILP using aux variables, just as in Tseitin transformation.

- $\underbrace{\underbrace{(A \wedge B)}_{P} \vee \underbrace{(A \wedge \sim\underbrace{(C \wedge \underbrace{(D \vee E)}_{Q})}_{R})}_{S}}_{T}$

Q >= D; Q >= E; Q <= D+E

R <= C; R <= Q; R >= C+Q-1

S <= A; S <= (1-R); S >= A+(1-R)-1

T >= P; T >= S; T <= P+S

P <= A; P <= B; P >= A+B-1

Finally, require T==1.
Or for a soft constraint,
add weight*T to the
maximization objective.

MAX-SAT example: Linear Ordering Problem

- Arrange these archaeological artifacts or fossils along a timeline
- Arrange a program's functions in a sequence so that callers tend to be above callees
- Poll humans based on *pairwise* preferences: Then sort the political candidates or policy options or acoustic stimuli into a *global* order
- In short:
  Sorting with a flaky comparison function
  - might not be asymmetric, transitive, etc.
  - can be weighted
    - the comparison "a < b" isn't boolean, but real
    - strongly positive/negative if we strongly want a to precede/follow b
  - maximize the sum of preferences
  - NP-hard

code thanks to Jason Smith

MAX-SAT example: Linear Ordering Problem

```
set X := { 1 … 50 };   # set of objects to be ordered
param G[X * X] := read "test.lop" as "<1n, 2n> 3n";

var LessThan[X * X] binary;
maximize goal: sum <x,y> in X * X : G[x,y] * LessThan[x,y];

subto irreflexive: forall <x> in X: LessThan[x,x] == 0;
subto antisymmetric_and_total: forall <x,y> in X * X with x < y:
    LessThan[x,y] + LessThan[y,x] == 1;   # what would <= and >= do?
subto transitive: forall <x,y,z> in X * X * X:   # if x<y and y<z then x<z
    LessThan[x,z] >= LessThan[x,y] + LessThan[y,z] - 1;

# alternatively (get this by adding LessThan[z,x] to both sides)
# subto transitive: forall <x,y,z> in X * X * X
#    with x < y and x < z and y != z:   # merely prevents redundancy
#    LessThan[x,y] + LessThan[y,z] + LessThan[z,x] <= 2; # no cycles
```

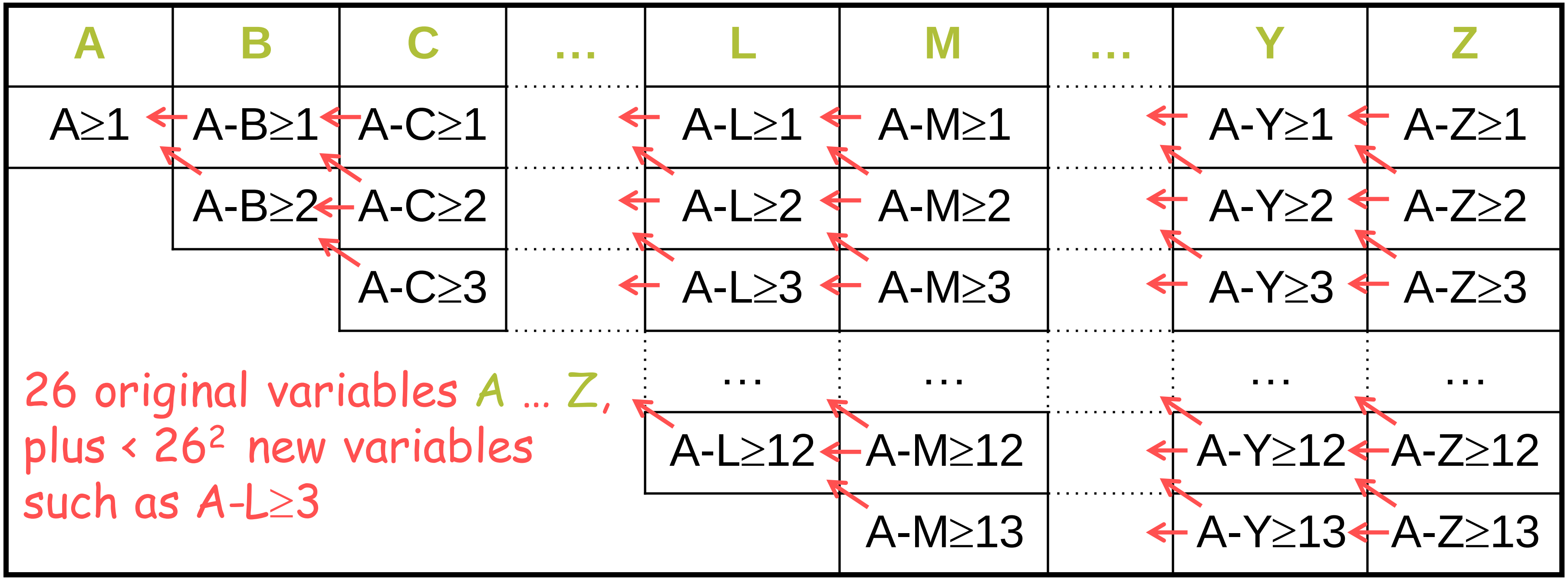
ZIMPL code thanks to Jason Smith

Why isn’t this just SAT all over again?

- Different solution techniques (we’ll compare)
- Much easier to encode “at least 13 of 26”:
  - Remember how we had to do it in pure SAT?

Encoding "at least 13 of 26"

(without listing all 38,754,732 subsets!)



A	B	C	…	L	M	…	Y	Z
A≥1	A-B≥1	A-C≥1		A-L≥1	A-M≥1		A-Y≥1	A-Z≥1
	A-B≥2	A-C≥2		A-L≥2	A-M≥2		A-Y≥2	A-Z≥2
		A-C≥3		A-L≥3	A-M≥3		A-Y≥3	A-Z≥3
				…	…		…	…
				A-L≥12	A-M≥12		A-Y≥12	A-Z≥12
					A-M≥13		A-Y≥13	A-Z≥13

26 original variables A … Z,
plus < 26^2 new variables
such as A-L≥3

- SAT formula should require that A-Z≥13 is true … and what else?
- yadayada ^ A-Z≥13 ^ (A-Z≥13 → (A-Y≥13 v (A-Y≥12 ^ Z)))
  ^ (A-Y≥13 → (A-X≥13 v (A-X≥12 ^ Y))) ^ …

one "only if" definitional constraint for each new variable

Why isn’t this just SAT all over again?

- Different solution techniques (we’ll compare)
- Much easier to encode “at least 13 of 26”:
  - a+b+c+…+z ≥ 13 (and solver exploits this)
  - Lower bounds on such sums are useful to model requirements
  - Upper bounds on such sums are useful to model limited resources
  - Can include real coefficients (e.g., c uses up 5.4 of the resource):
    - a + 2b + 5.4c + … + 0.3z ≥ 13 (very hard to express with SAT)
    - MAX-SAT allows an overall soft constraint, but not a limit of 13 (nor a piecewise-linear penalty function for deviations from 13)
- Mixed integer programming combines the power of SAT and disjunction with the power of numeric constraints
  - Even if some variables are boolean, others may be integer or real and constrained by linear equations (“Mixed Integer Programming”)

Logical control of real-valued constraints

- Want $\delta=1$ to force an inequality constraint to turn on: (where δ is a binary variable)
- Idea: $\delta=1 \rightarrow a\cdot x \le b$
- Implementation: $a\cdot x \le b+M(1-\delta)$ where M very large
  - Requires $a\cdot x \le b+M$ always, so set M to upper bound on $a\cdot x - b$

- Conversely, want satisfying the constraint to force $\delta=1$:
- Idea: $a\cdot x \le b \rightarrow \delta=1$ or equivalently $\delta=0 \rightarrow a\cdot x > b$
- Implementation:
  - approximate by $\delta=0 \rightarrow a\cdot x \ge b+0.001$
  - implement as $a\cdot x + \text{surplus}*\delta \ge b+0.001$
  - more precisely $a\cdot x \ge b+0.001 + (m-0.001)*\delta$ where m very negative
    - Requires $a\cdot x \ge b+m$ always, so set m to lower bound on $a\cdot x - b$

Logical control of real-valued constraints

- If some inequalities hold, want to enforce others too.
- ZIMPL doesn't (yet?) let us write
  - subto foo: (a.x <= b and c.x <= d) --> (e.x <= f or g.x <= h)

  but we can manually link these inequalities to binary variables:
  - $a.x \leq b \rightarrow \delta_1$ implement as on bottom half of previous slide
  - $c.x \leq d \rightarrow \delta_2$ implement as on bottom half of previous slide
  - $(\delta_1$ and $\delta_2) \rightarrow \delta_3$ implement as $\delta_3 \geq \delta_1 + \delta_2 - 1$
  - $\delta_3 \rightarrow (\delta_4$ or $\delta_5)$ implement as $\delta_3 \leq \delta_4 + \delta_5$
  - $\delta_4 \rightarrow e.x \leq f$ implement as on top half of previous slide
  - $\delta_5 \rightarrow g.x \leq h$ implement as on top half of previous slide

- Partial shortcut in ZIMPL using "vif … then … else .. end" construction:
  - subto foo1: vif (δ_1==0) then a.x >= b+0.001 end;
  - subto foo2: vif (δ_2==0) then c.x >= d+0.001 end;
  - subto foo3: vif ((δ_1==1 and δ_2==1) and not (δ_4==1 or δ_5==1))
    then $\delta_1 \geq \delta_1$+1 end; # i.e., the "vif" condition is impossible
  - subto foo4: vif (δ_4==1) then e.x <= f end;
  - subto foo5: vif (δ_5==1) then g.x <= h end;

Integer programming beyond 0-1:
N-Queens Problem

- param queens := 8;
- set C := {1 .. queens};
- var row[C] integer >= 1 <= queens;

- set Pairs := {<i,j> in C*C with i < j}; i < j to avoid duplicate constraints
- subto alldifferent: forall <i,j> in Pairs: row[i] != row[j];
- subto nodiagonal: forall <i,j> in Pairs: vabs(row[i]-row[j]) != j-i;
- # no line saying what to maximize or minimize

Instead of writing x != y in ZIMPL, or (x-y) != 0,
need to write vabs(x-y) >= 1. (if x,y integer; what if they're real?)

This is equivalent to v >= 1 where v is forced (how?) to equal |x-y|.

v >= x-y, v >= y-x, and add v to the minimization objective.

No, can't be right def of v: LP alone can't define non-convex feasible region.

And it <u>is</u> wrong: this encoding will <u>allow</u> x==y and just choose v=1 anyway!

Correct solution: use ILP. Binary var δ, with $\delta=0 \rightarrow v=x-y$, $\delta=1 \rightarrow v=y-x$.

Or more simply, eliminate v: $\delta=0 \rightarrow x-y \geq 1$, $\delta=1 \rightarrow y-x \geq 1$.

program example from ZIMPL manual

Integer programming beyond 0-1: Allocating Indivisible Objects

- Airline scheduling
  (can’t take a fractional number of passengers)
- Job shop scheduling (like homework 2)
  (from a set of identical jobs, each machine takes an integer #)
- Knapsack problems (like homework 4)
- Others?

Harder Real-World Examples of LP/ILP/MIP

Unsupervised Learning of a Part-of-Speech Tagger

- based on Ravi & Knight 2009

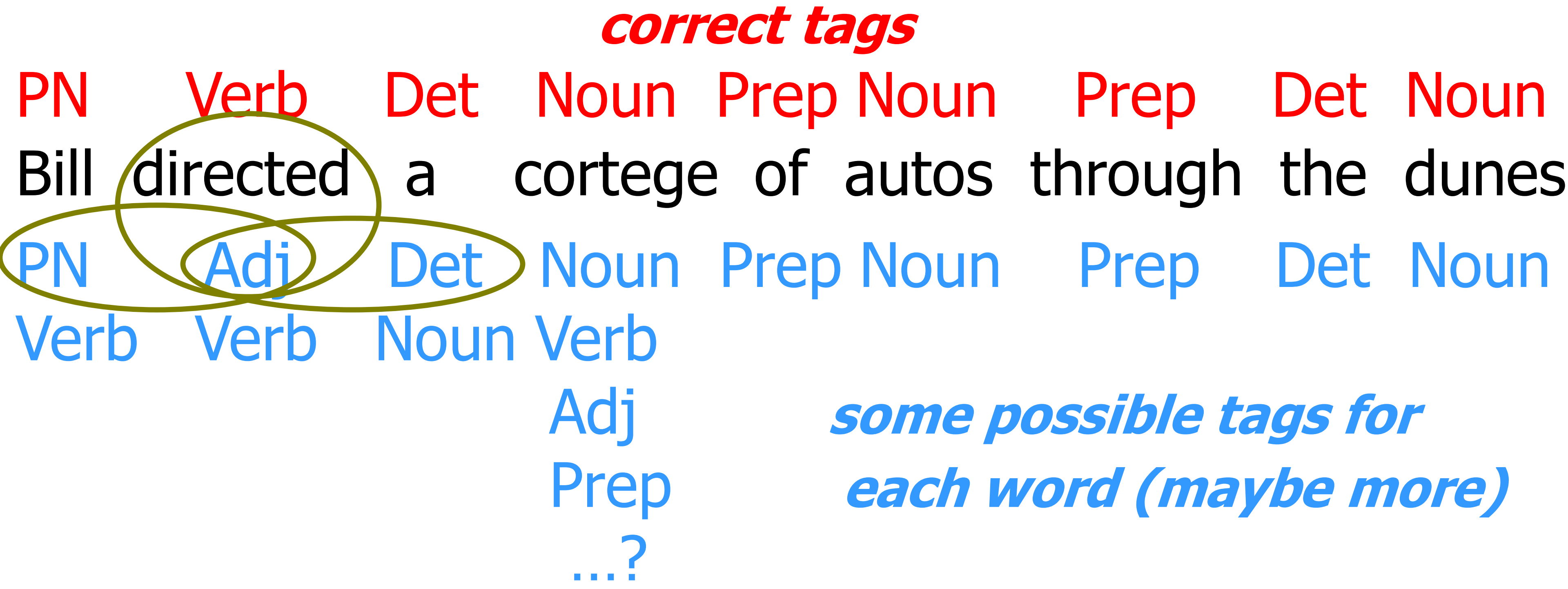
Part-of-speech tagging

Input: `the lead paint is unsafe`

Output: `the/Det lead/N paint/N is/V unsafe/Adj`

- Partly supervised learning:
  - You have a lot of text (without tags)
  - You have a dictionary giving possible tags for each word

What Should We Look At?



correct tags

PN	Verb	Det	Noun	Prep	Noun	Prep	Det	Noun
Bill	directed	a	cortege	of	autos	through	the	dunes
PN	Adj	Det	Noun	Prep	Noun	Prep	Det	Noun
Verb	Verb	Noun	Verb					
			Adj					
			Prep					
			…?					

some possible tags for each word (maybe more)

Each unknown tag is **constrained** by its word and by the tags to its immediate left and right. But those tags are unknown too …

What Should We Look At?

correct tags

PN	Verb	Det	Noun	Prep	Noun	Prep	Det	Noun
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Verb	Verb	Noun	Verb					
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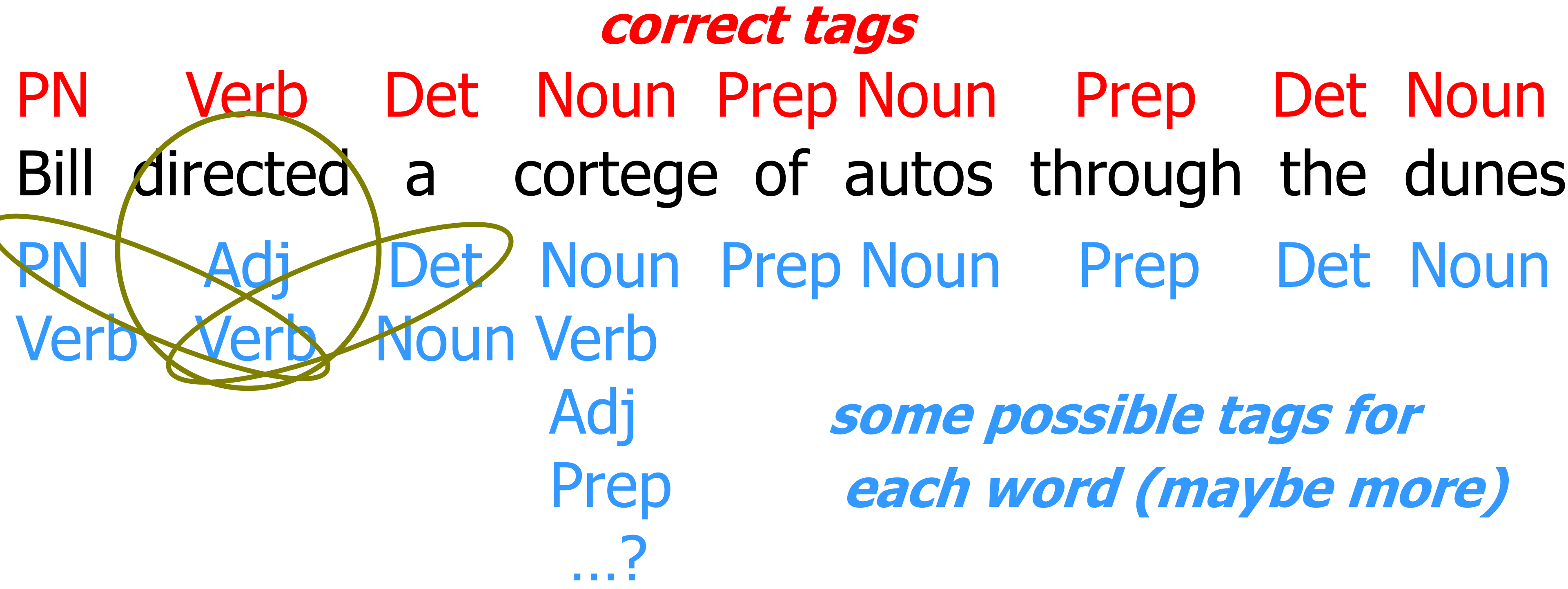

correct tags

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Verb	Verb	Noun	Verb					
			Adj					
			Prep					
			…?					

some possible tags for each word (maybe more)

Each unknown tag is **constrained** by its word and by the tags to its immediate left and right. But those tags are unknown too …

Unsupervised Learning of a Part-of-Speech Tagger

- Given k tags (Noun, Verb, ...)
- Given a dictionary of m word types (aardvark, abacus, …)
- Given some text: n word tokens (The aardvark jumps over…)
- Want to pick: n tags (Det Noun Verb Prep..)

- Encoding as variables?
- How to inject some knowledge about types and tokens?
- Constraints and objective?
  - Few tags allowed per word
  - Few 2-tag sequences allowed (e.g., "Det Det" is bad)
  - Tags may be correlated with one another, or with word endings

Minimum spanning tree ++

- based on Martins et al. 2009

Traveling Salesperson

- Version with subtour elimination constraints
- Version with auxiliary variables