Constraint Program Solvers

Generalize SAT solvers

- Try to generalize systematic SAT solvers.
  - Note: Straightforward to generalize the stochastic ones.
- Recall SAT enhancements to backtracking search:
  - Careful variable ordering
  - When we instantiate a var, shorten other clauses
    - May detect conflicts
    - May result in unit clauses, instantiating other vars: can propagate those immediately ("unit propagation")
  - Conflict analysis when forced to backtrack
    - Backjumping
    - Clause learning
    - Improved variable ordering

Andrew Moore's animations

Graph coloring: Color every vertex so that adjacent vertices have different colors.

(NP-complete, many applications such as register allocation)

http://www-2.cs.cmu.edu/~awm/animations/constraint/

A few of the many propagation techniques

Simple backtracking

All search, no propagation

Good tradeoff for your problem?

All propagation, no search (propagate first, then do "backtrack-free" search)

Adaptive consistency (variable elimination)

A few of the many propagation techniques

Simple backtracking (no propagation)

commonly chosen

Forward checking (reduces domains)

Arc consistency (reduces domains & propagates)
(limited versions include unit propagation, bounds propagation)

i-consistency (fuses constraints)

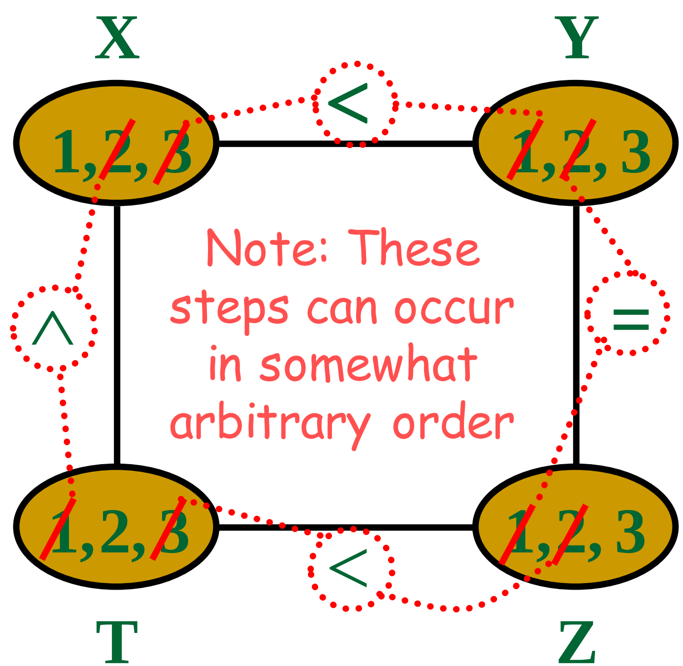
Adaptive consistency (variable elimination)

Arc consistency (= 2-consistency)

This example is more interesting than graph coloring or SAT:

- The < constraint (unlike ≠ in graph coloring) allows propagation before we know any var's exact value. X must be < some Y, so can't be 3.
- Hence we can propagate before we start backtracking search.
- (In SAT, we could only do that if original problem had unit clauses.)

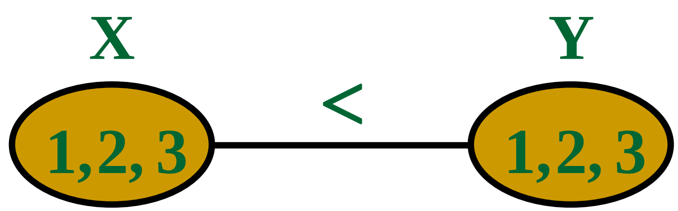
X, Y, Z, T :: 1..3
X #< Y
Y #= Z
T #< Z
X #< T

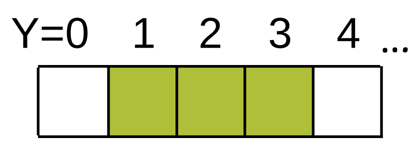


propagation completely solved the problem! No further search necessary (this time).

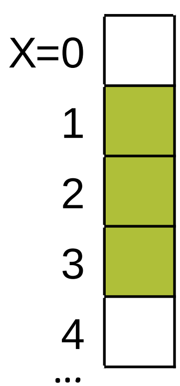
slide thanks to Rina Dechter (modified)

Arc consistency is the result of "joining" binary and unary constraints

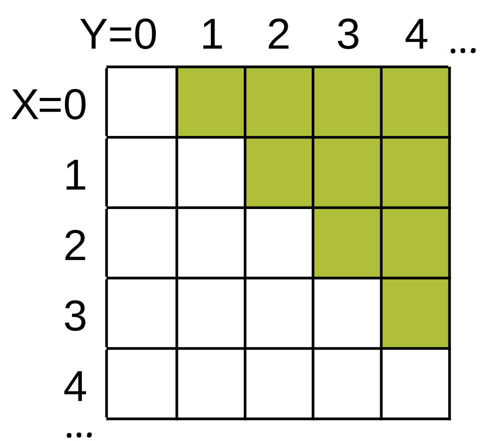




Y::1..3



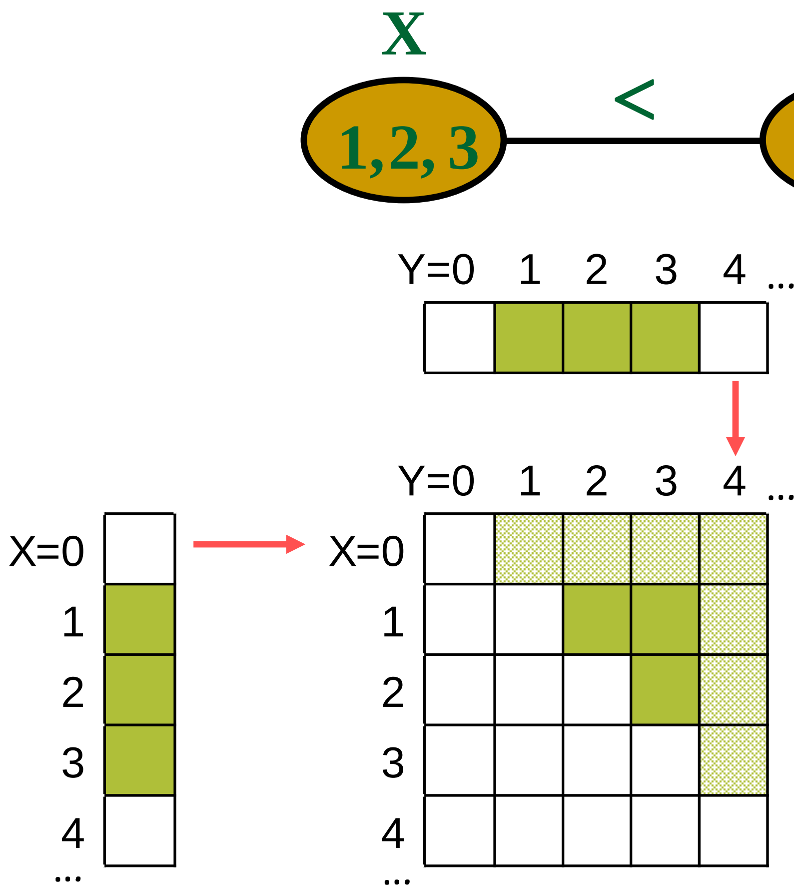
X::1..3



X #< Y

(X #< Y is an infinite table of possible vals, not shown in full)

Arc consistency is the result of "joining" binary and unary constraints



Y::1..3

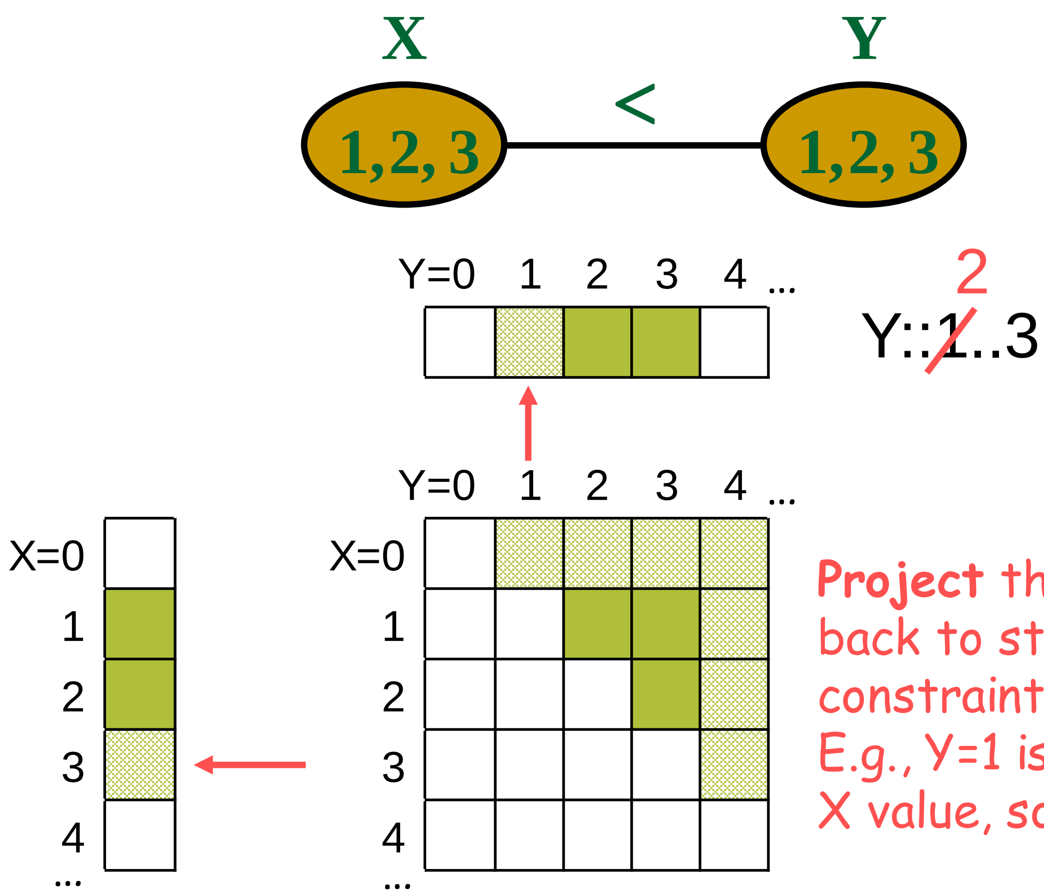
really a database join of the 3 constraints, on columns X and Y (their common vars)

Create table of all (X,Y) that are consistent among all 3 constraints

X::1..3

X #< Y **and** X::1..3 **and** Y::1..3

Arc consistency is the result of "joining" binary and unary constraints



These inferred restrictions to X, Y now propagate further through other constraints …

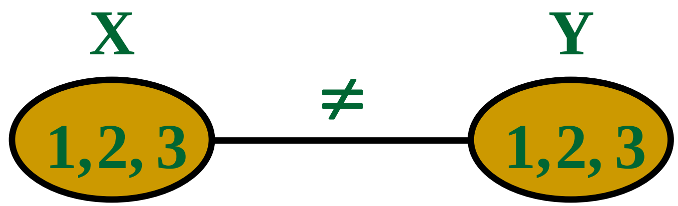
Project those restricted values back to strengthen the unary constraints.
E.g., Y=1 is not consistent with **any** X value, so kill it off.

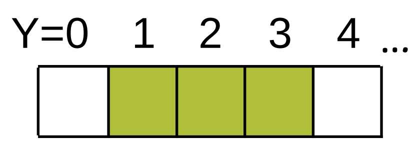
X::1..~~3~~ 2

X #< Y **and** X::1..3 **and** Y::1..3

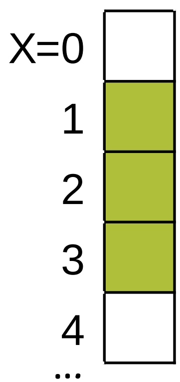
Another example: Graph coloring

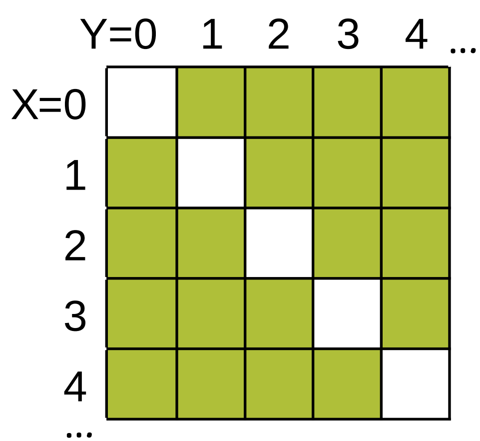
(1,2,3 = blue, red, black)





Y::1..3



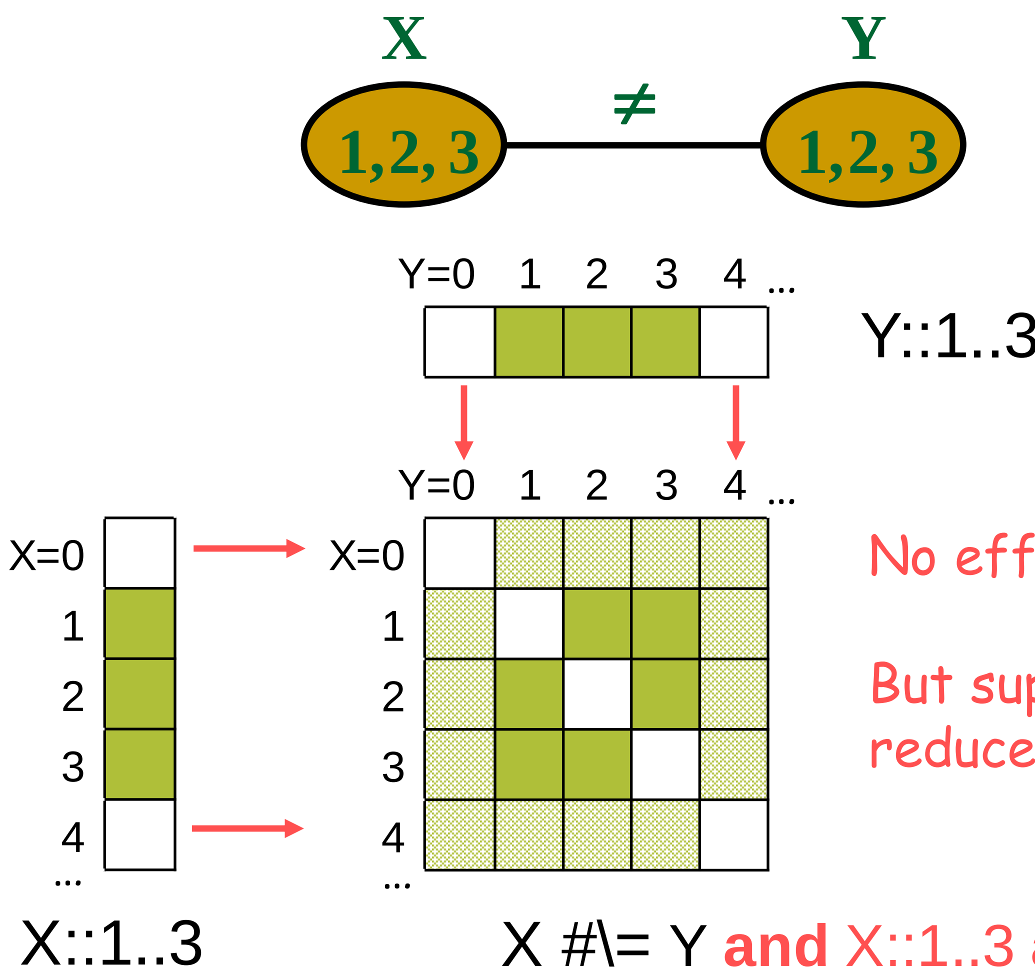


X::1..3

X #\= Y

Another example: Graph coloring

(1,2,3 = blue, red, black)



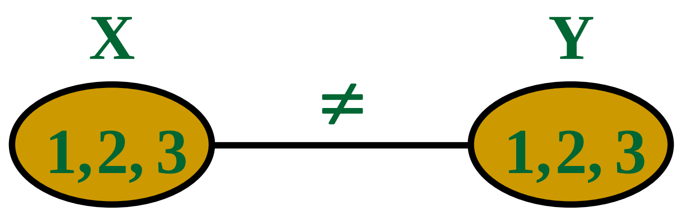
No effect yet on X, Y domains.

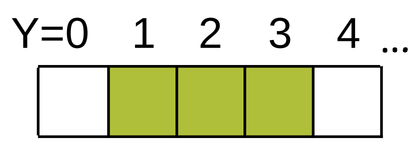
But suppose **other** propagations reduce X's domain to 2 …

X #\= Y **and** X::1..3 **and** Y::1..3

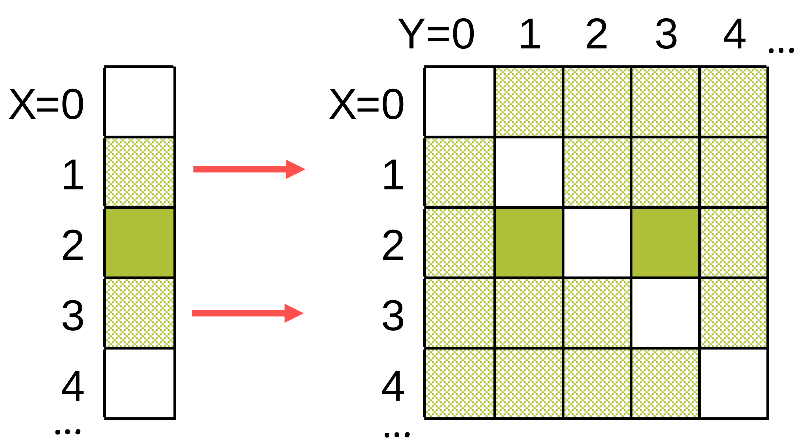
Another example: Graph coloring

(1,2,3 = blue, red, black)





Y::1..3



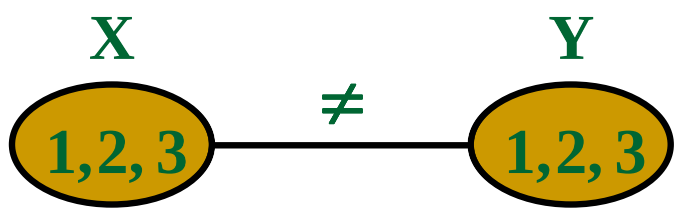
But suppose **other** propagations now reduce X's domain to 2 …

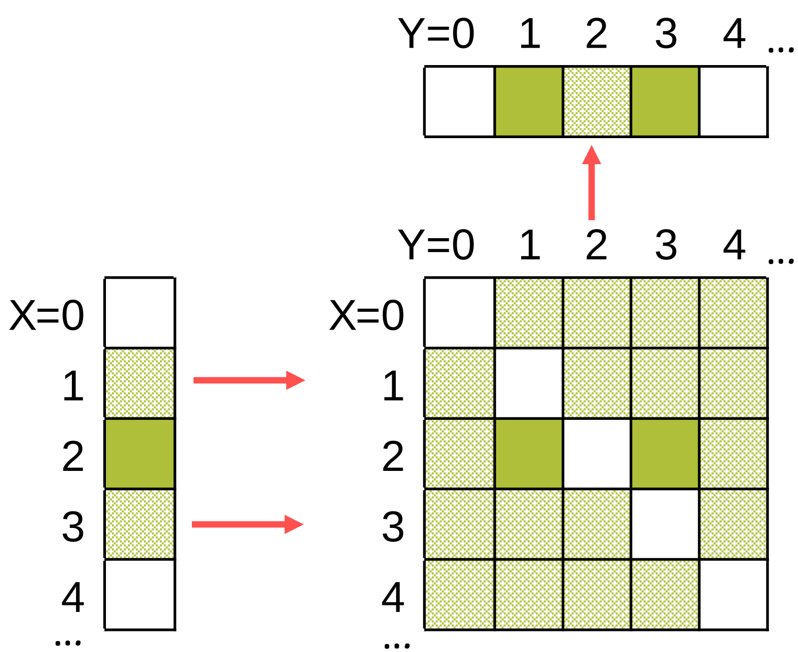
X::2

X #\= Y **and** X::2 **and** Y::1..3

Another example: Graph coloring

(1,2,3 = blue, red, black)





Y::~~1..3~~ [1,3]

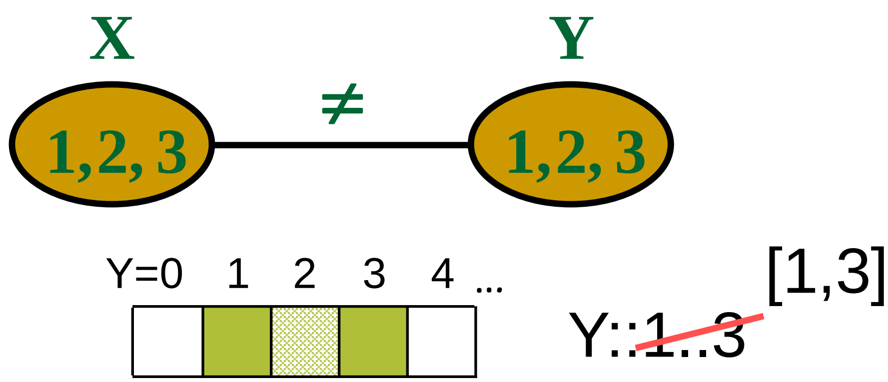
But suppose **other** propagations reduce X's domain to 2 … we find Y ≠ 2.

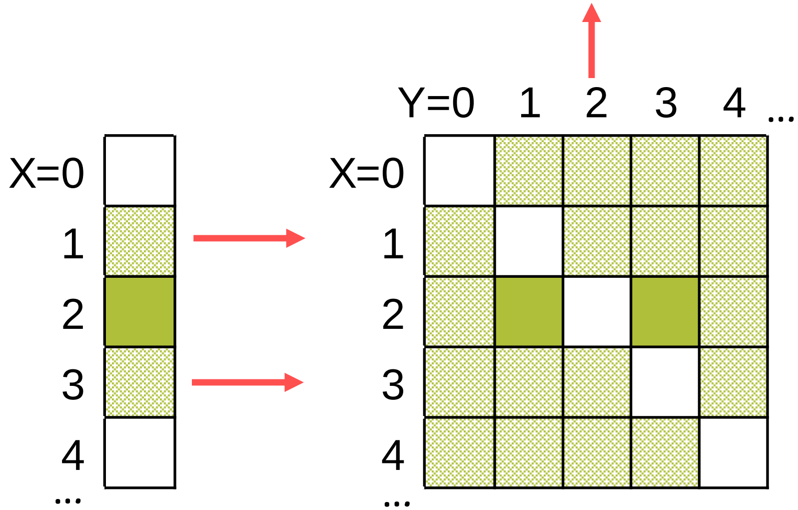
X::2

X #\= Y **and** X::2 **and** Y::1..3

Another example: Graph coloring

(1,2,3 = blue, red, black)





Standard algorithm "AC-3":
Whenever X changes, construct this (X,Y) grid and see what values of Y appear on at least one green square.
Here only Y=1 and Y=3 appear.

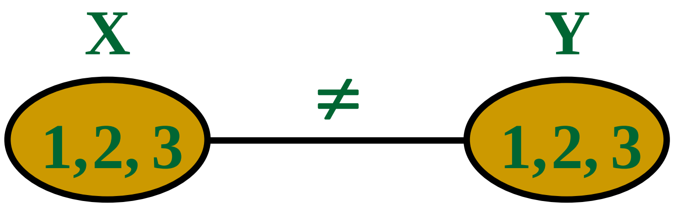
X::2

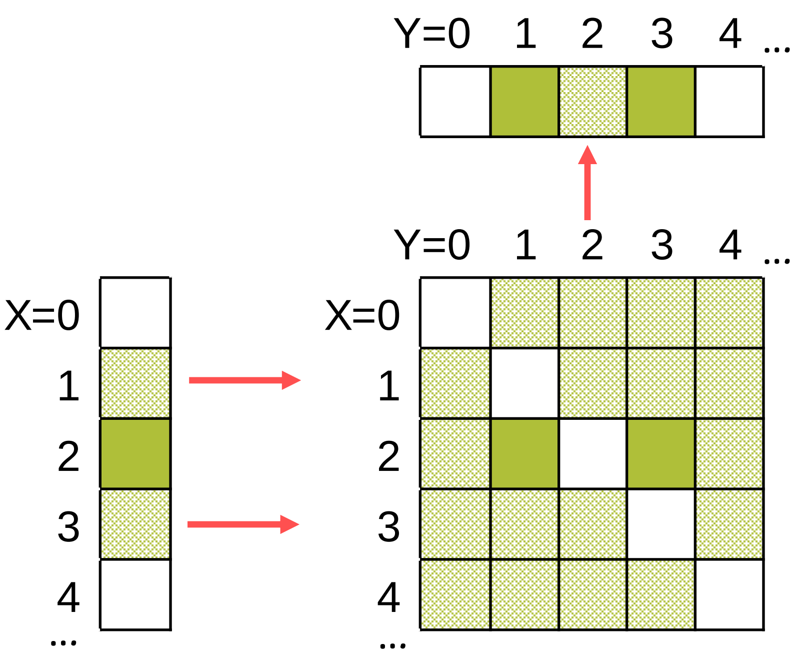
X #\= Y **and X::2 and Y::1..3**

Another example: Graph coloring

(1,2,3 = blue, red, black)

AC-3: if X's domain changes, recompute Y's domain from scratch ("variable granularity")
AC-4: if X's domain loses a particular value, reduce support for particular values in Y's domain ("value granularity")





[1,3]
Y::~~1..3~~

Theoretically more efficient algorithm "AC-4": Maintain the grid. Remember how many green squares are in the Y=2 column. When this counter goes to 0, conclude Y≠2.

X::2

X #\= Y **and** X::2 **and** Y::1..3

(there's some recent work on speeding this up with the "watched variable" trick here)

Another example: Simplified magic square

- Ordinary magic square uses alldifferent numbers 1..9
- But for simplicity, let's allow each var to be 1..3

V_1	V_2	V_3	This row must sum to 6
V_4	V_5	V_6	This row must sum to 6
V_7	V_8	V_9	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Another example: Simplified magic square

- [V1, V2, … V9] :: 1..3
- V1 + V2 + V3 #= 6, etc.

Not actually a binary constraint; basically we can keep our algorithm, but it's now called "generalized" arc consistency.

V_1	V_2	V_3	This row must sum to 6
V_4	V_5	V_6	This row must sum to 6
V_7	V_8	V_9	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Propagate on Semi-magic Square

- No propagation possible yet
- So start backtracking search here

123	123	123	This row must sum to 6
123	123	123	This row must sum to 6
123	123	123	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Propagate on Semi-magic Square

- So start backtracking search here

now generalized arc consistency kicks in!

1	123	123	This row must sum to 6
123	123	123	This row must sum to 6
123	123	123	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Propagate on Semi-magic Square

- So start backtracking search here

any further propagation from these changes?

1	23	23	This row must sum to 6
23	23	123	This row must sum to 6
23	123	23	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Propagate on Semi-magic Square

- So start backtracking search here

any further propagation from these changes? yes …

1	23	23	This row must sum to 6
23	23	12	This row must sum to 6
23	12	23	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Propagate on Semi-magic Square

- So start backtracking search here
  - That's as far as we can propagate, so try choosing a value here

1	2	23	This row must sum to 6
23	23	12	This row must sum to 6
23	12	23	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Propagate on Semi-magic Square

- So start backtracking search here
  - That's as far as we can propagate, so try choosing a value here … more propagation kicks in!

1	2	3	This row must sum to 6
2	3	1	This row must sum to 6
3	1	2	This row must sum to 6
This column must sum to 6	This column must sum to 6	This column must sum to 6	This diagonal must sum to 6

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

Search tree with propagation

123	123	123
123	123	123
123	123	123

1	23	23
23	23	12
23	12	23

2	123	123
123	123	123
123	123	123

3	12	12
12	12	23
12	23	12

1	2	3
2	3	1
3	1	2

1	3	2
3	2	1
2	1	3

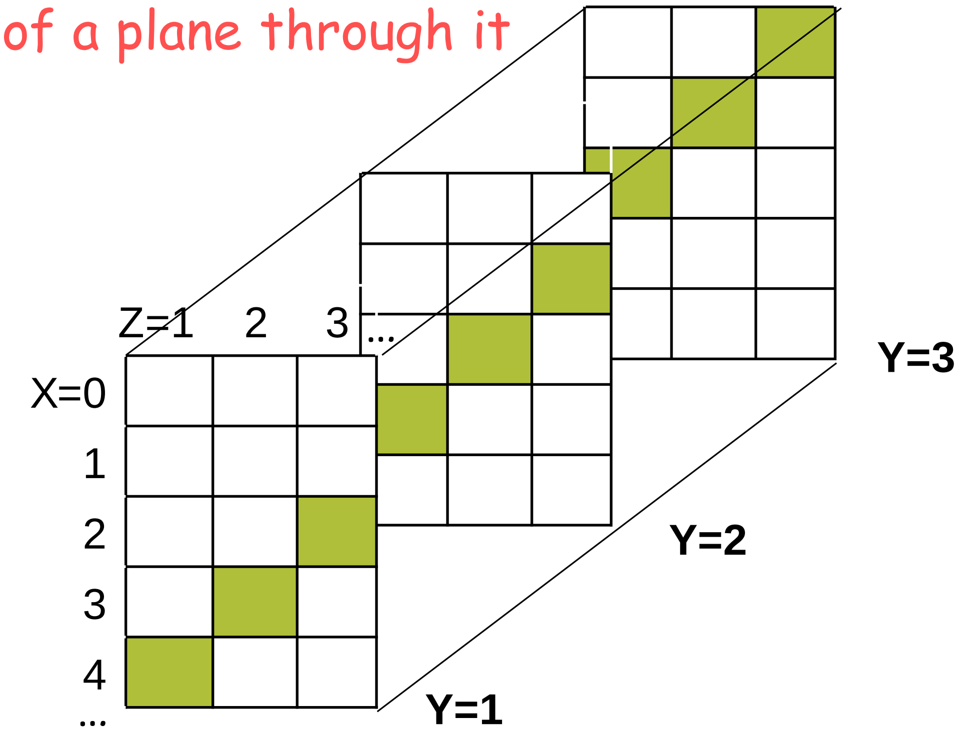
In fact, we never have to consider these if we stop at first success

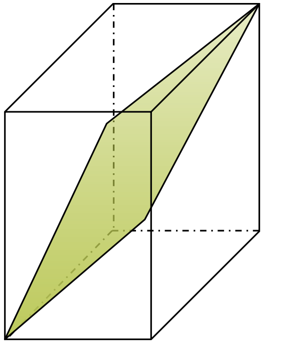
slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

So how did generalized arc consistency (non-binary constraint) work just now?

a cube of (X,Y,Z) triples

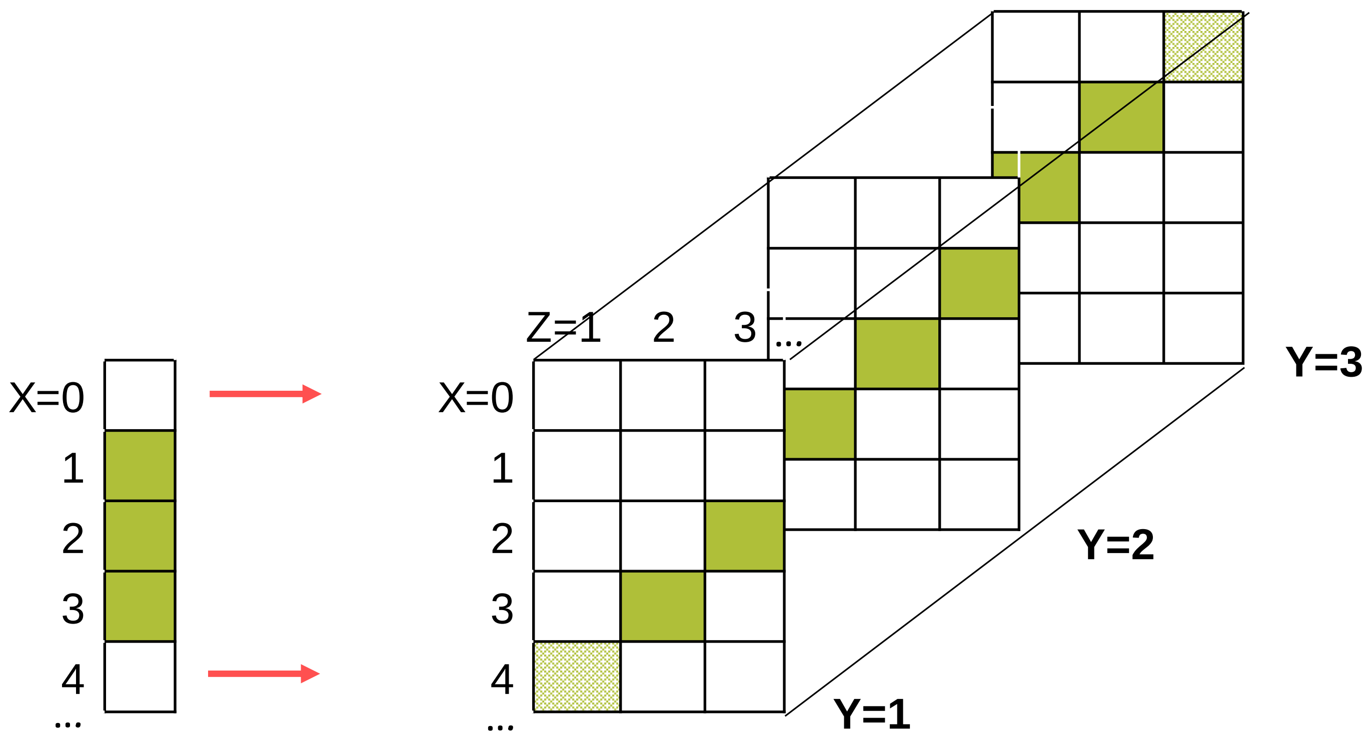
X+Y+Z=6 is equation of a plane through it





X + Y + Z #= 6

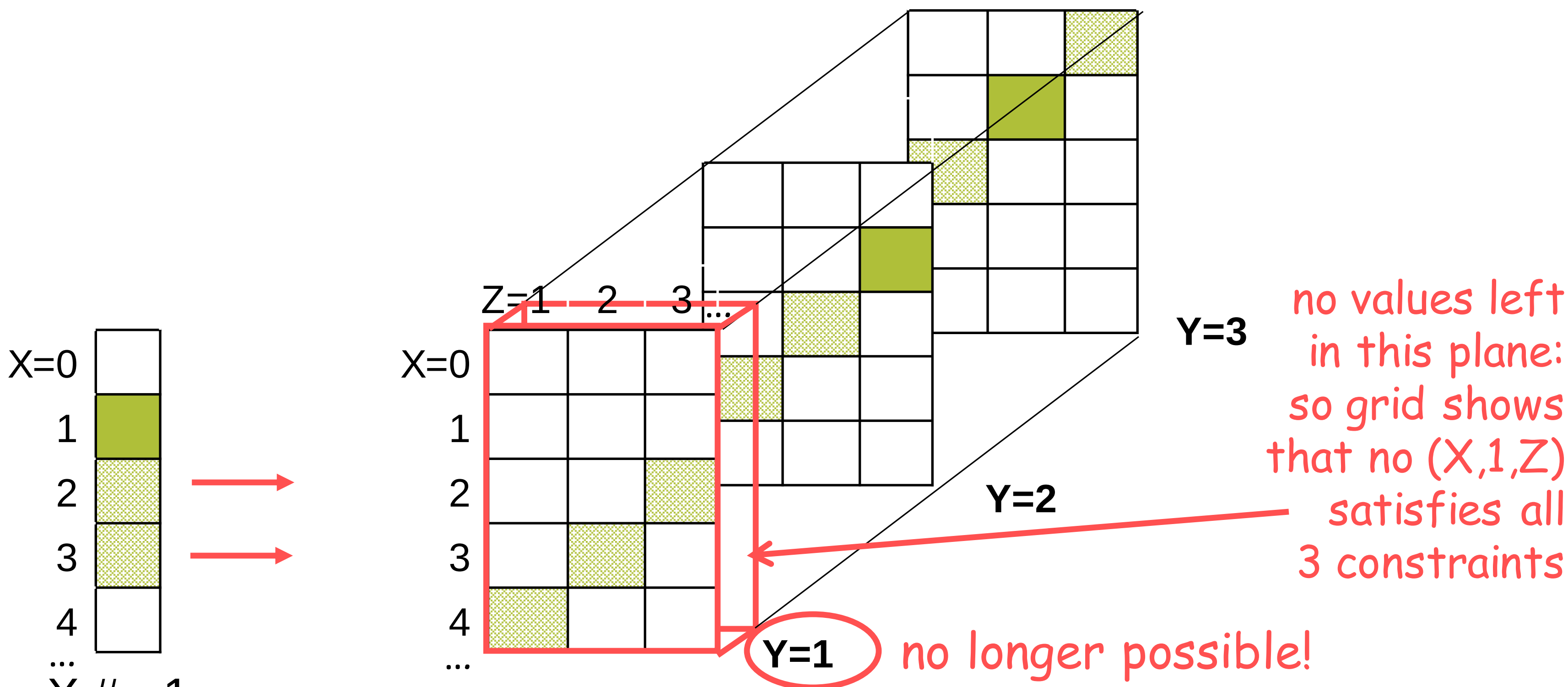
So how did generalized arc consistency (non-binary constraint) work just now?



X::1..3

X + Y + Z #= 6
and [X,Y,Z]::1..3

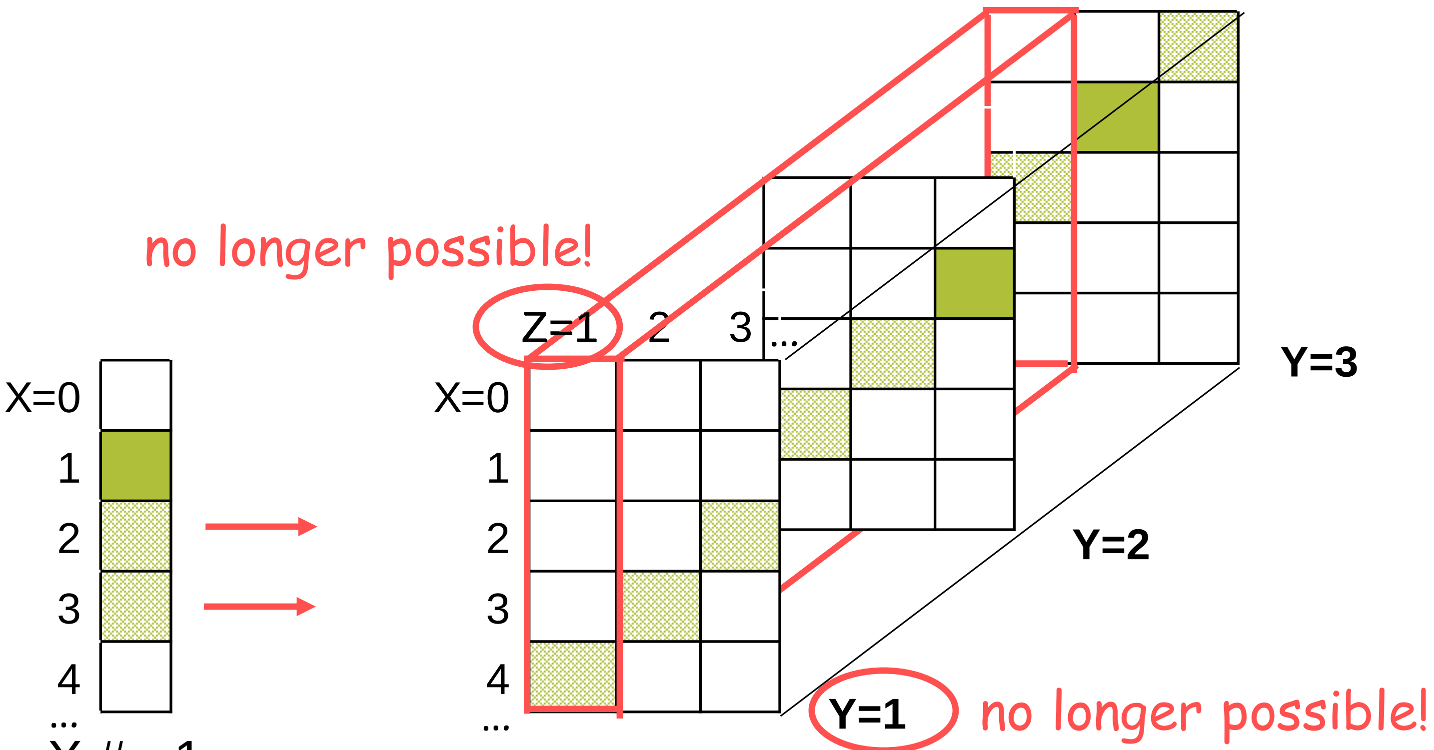
So how did generalized arc consistency (non-binary constraint) work just now?



X #= 1 (assigned during search)

X + Y + Z #= 6
and X #= 1 **and** [Y,Z]::1..3

So how did generalized arc consistency (non-binary constraint) work just now?

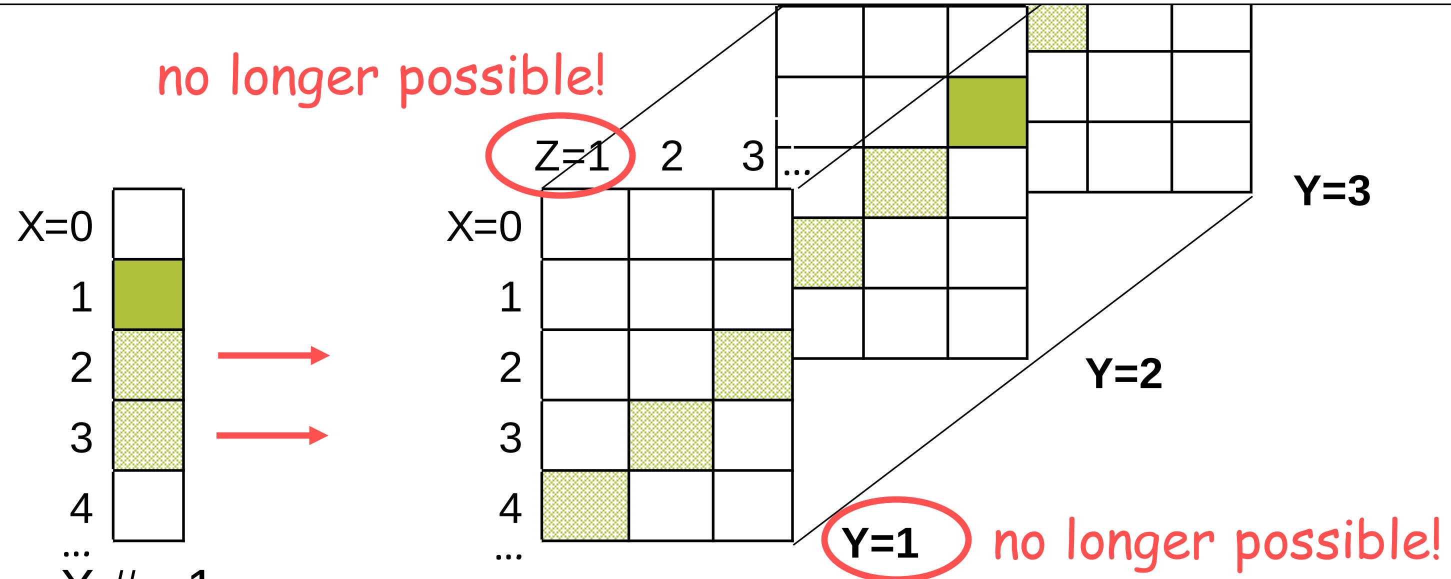


X #= 1
(assigned during search)

X + Y + Z #= 6
and X #= 1 **and** [Y,Z]::1..3

How do we compute the new domains in practice?

AC-3 algorithm from before:
Nested loop over all (X,Y,Z) triples with X#=1, Y::1..3, Z::1..3
See which ones satisfy X+Y+Z #= 6 (green triples)
Remember which values of Y, Z **occurred** in green triples

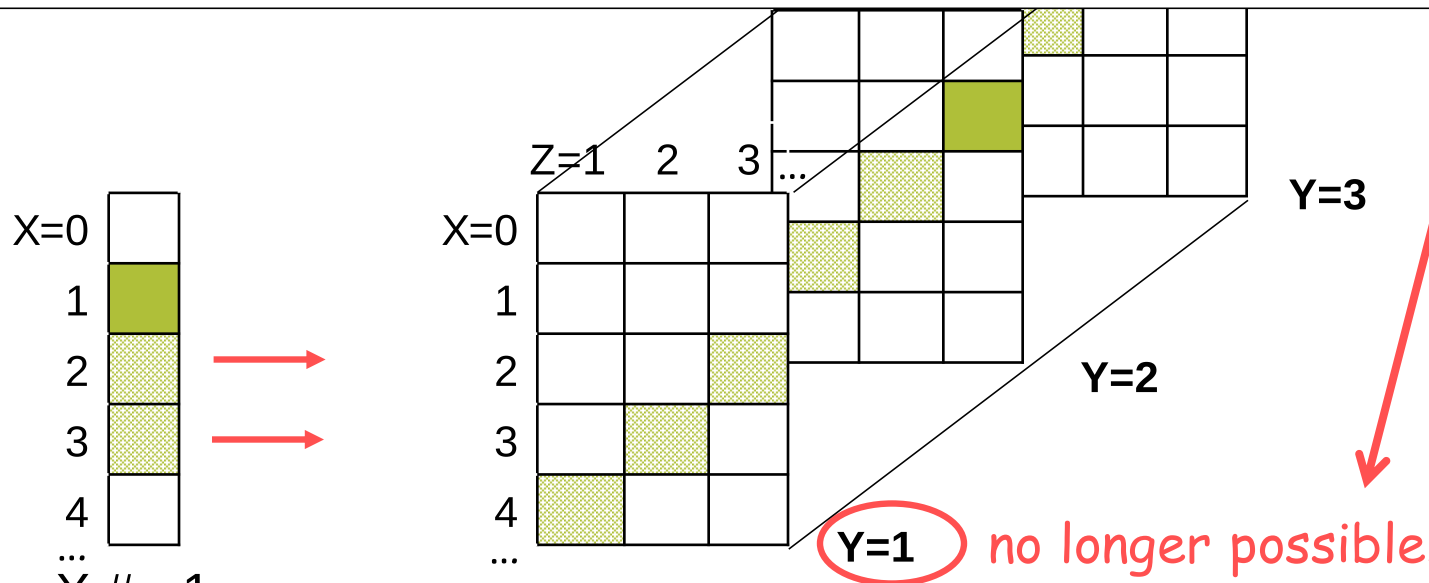


X #= 1
(assigned during search)

X + Y + Z #= 6
and X #= 1 **and** [Y,Z]::1..3

How do we compute the new domains in practice?

Another option: Reason about the constraints symbolically!
X #= 1 **and** X+Y+Z #= 6 ➔ 1+Y+Z #= 6 ➔ Y+Z #= 5
We inferred a new constraint!
Use it to reason further: Y+Z #=5 **and** Z #<= 3 ➔ Y #>= 2

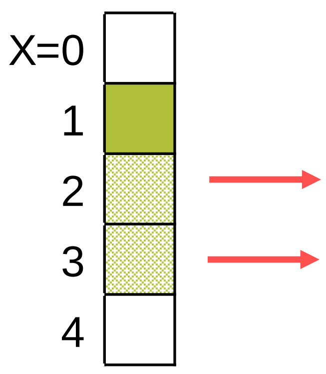


X #= 1
(assigned during search)

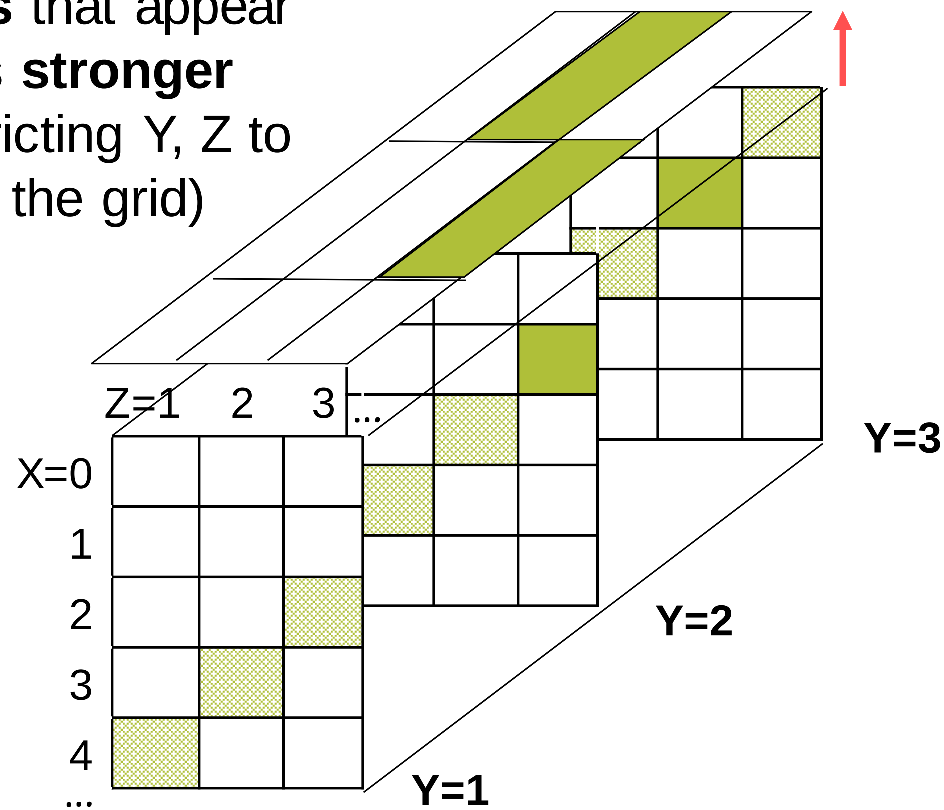
X + Y + Z #= 6
and X #= 1 **and** [Y,Z]::1..3

How do we compute the new domains in practice?

Our inferred constraint Y + Z #= 5 restricts (Y,Z) to **pairs** that appear in the 3-D grid (this is **stronger** than *individually* restricting Y, Z to values that appear in the grid)



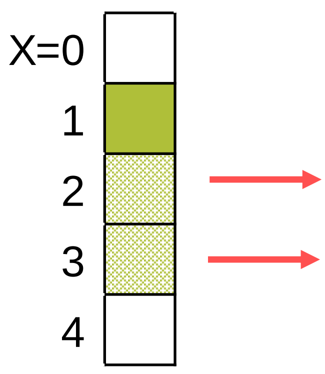
X #= 1 (assigned during search)



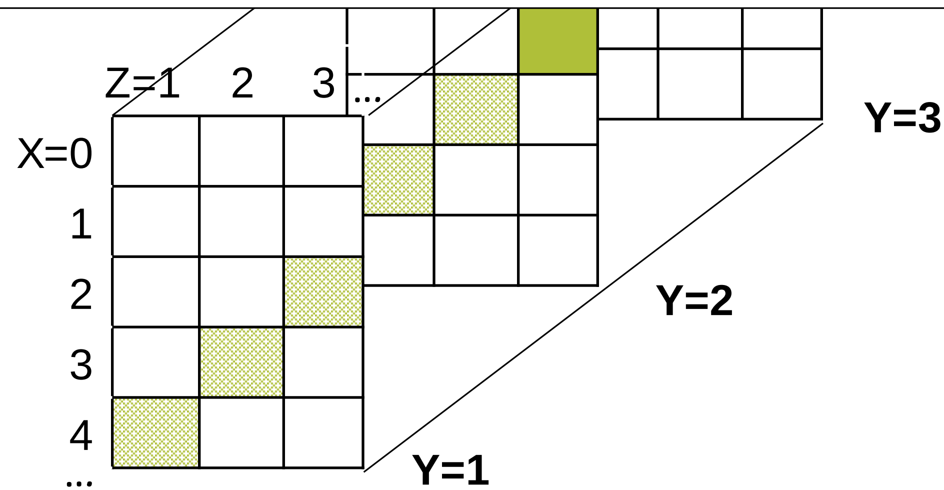
X + Y + Z #= 6
and X #= 1 **and** [Y,Z]::1..3

How do we compute the new domains in practice?

Another option: Reason about the constraints symbolically!
X #= 1 **and** X+Y+Z #= 6 ➔ 1+Y+Z #= 6 ➔ Y+Z #= 5
That's exactly what we did for SAT:
~X **and** (X v Y v ~Z v W) ➔ (Y v ~Z v W)
(we didn't loop over all values of Y, Z, W to figure this out)



X #= 1 (assigned during search)



X + Y + Z #= 6
and X #= 1 **and** [Y,Z]::1..3

How do we compute the new domains in practice?

Another option: Reason about the constraints symbolically!
X #= 1 **and** X+Y+Z #= 6 ➔ 1+Y+Z #= 6 ➔ Y+Z #= 5
That's exactly what we did for SAT:
~X **and** (X v Y v ~Z v W) ➔ (Y v ~Z v W)

- Symbolic reasoning can be more efficient:
  X #< 40 **and** X+Y #= 100 ➔ Y #> 60
  (vs. iterating over a large number of (X,Y) pairs)
- But requires the solver to know stuff like algebra!
- Use "constraint handling rules" to symbolically propagate changes in var domains through particular types of constraints
  - E.g., linear constraints: 5*X + 3*Y - 8*Z #>= 75
  - E.g., boolean constraints: X v Y v ~Z v W
  - E.g., alldifferent constraints: alldifferent(X,Y,Z) – come back to this

Strong and weak propagators

- Designing good propagators (constraint handling rules)
  - a lot of the "art" of constraint solving
  - the subject of the rest of the lecture
- Weak propagators run fast, but may not eliminate all impossible values from the variable domains.
  - So backtracking search must consider & eliminate more values.
- Strong propagators work harder – not always worth it.

- Use "constraint handling rules" to symbolically propagate changes in var domains through particular types of constraints
  - E.g., linear constraints: 5*X + 3*Y - 8*Z #>= 75
  - E.g., boolean constraints: X v Y v ~Z v W
  - E.g., alldifferent constraints: alldifferent(X,Y,Z) – come back to this

Example of weak propagators: Bounds propagation for linear constraints

- [A,B,C,D] :: 1..100
- A #\= B (inequality)
- B + C #= 100
- 7*B + 3*D #> 50

Might want to use a simple, weak propagator for these linear constraints: Revise C, D **only** if something changes B's **total range** min(B)..max(B).

If we learn that B $\geq$ x, for some const x,
conclude C $\leq$ 100-x
If we learn that B $\leq$ y,
conclude C $\geq$ 100-y
and $\boxed{D > (50+7y)/3}$

So new lower/upper bounds on B give new bounds on C, D. That is, shrinking B's range shrinks other variables' ranges.

	B	$\leq$ y
Multiply:	-7B	$\geq$ -7y
Add to:	7B + 3D > 50	
Result:	3D > 50-7y	
Therefore:	D > (50-7y)/3	

Example of weak propagators: Bounds propagation for linear constraints

- [A,B,C,D] :: 1..100
- A #\= B (inequality)
- B + C #= 100
- 7*B + 3*D #> 50

Might want to use a simple, weak propagator for these linear constraints: Revise C, D **only** if something changes B's **total range** min(B)..max(B).
Why is this only a weak propagator?
It does nothing if B gets a hole in the middle of its range.

Suppose we discover or guess that A=75
Full arc consistency would propagate as follows:

domain(A) changed – revise B :: ~~1..100~~ [1..74, 76..100]
domain(B) changed – revise C :: ~~1..100~~ [1..24, 26..100]
domain(B) changed – revise D :: ~~1..100~~ 1..100
(wasted time figuring out there was no change)

our bounds propagator doesn't try to get these revisions.

Bounds propagation can be pretty powerful …

- sqr(X) $= 7-X   (remember #= for integers, $= for real nums)
  - Two solutions: $\frac{-1 \pm \sqrt{29}}{2} \approx \{2.193, -3.193\}$
- ECLiPSe internally introduces a variable Y for the intermediate quantity sqr(X):
  - Y $= sqr(X)   hence Y $>= 0 by a rule for sqr constraints
  - Y $= 7-X   hence X $<= 7 by bounds propagation
  - That's all the propagation, so must do backtracking search.
    - We could try X=3.14 as usual by adding new constraint X$=3.14
    - But we can't try each value of X in turn – too many options!
    - So do **domain splitting**: try X $>= 0, then X $< 0.
    - Now bounds propagation homes in on the solution! (next slide)

Bounds propagation can be pretty powerful …

- Y $= sqr(X)   hence Y ≥ 0
- Y $= 7-X   hence X ≤ 7 by bounds propagation
- X $>= 0   assumed by domain splitting during search

hence Y ≤ 7 by bounds propagation on Y $= 7-X

hence X ≤ 2.646 by bounds prop. on Y $= sqr(X)

(using a rule for sqr that knows how to take sqrt)

hence Y ≥ 4.354 by bounds prop. on Y $= 7-X

hence X ≥ 2.087 by bounds prop. on Y $= sqr(X)
(since we already have X ≥ 0)

hence Y ≤ 4.913 by bounds prop. on Y $= 7-X

hence X ≤ 2.217 by bounds prop. on Y $= sqr(X)

hence Y ≥ 4.783 by bounds prop. on Y $= 7-X

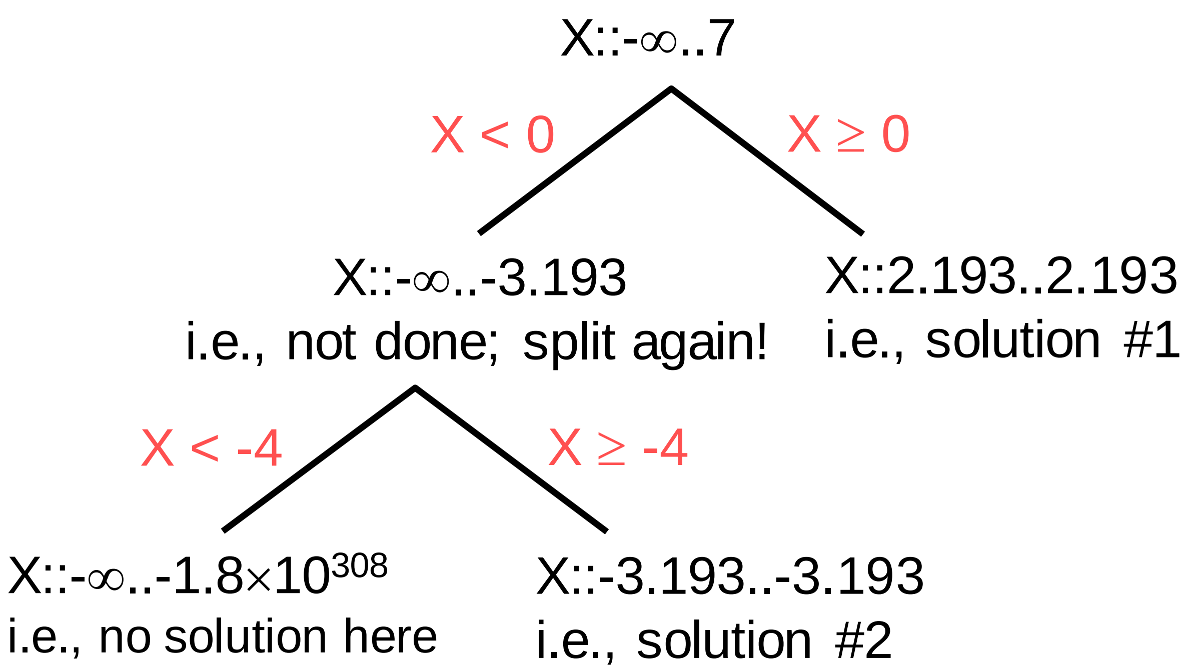
hence X ≥ 2.187 by bounds prop. on Y $= sqr(X)
(since we already have X ≥ 0)

- At this point we've got X :: 2.187 .. 2.217
- Continuing will narrow in on X = 2.193 by propagation alone!

Bounds propagation can be pretty powerful …

- Y $= sqr(X),
  Y $= 7-X,
  locate([X], 0.001). % like "labeling" for real vars;
  % 0.001 is precision for how finely to split domain

- Full search tree with (arbitrary) domain splitting and propagation:



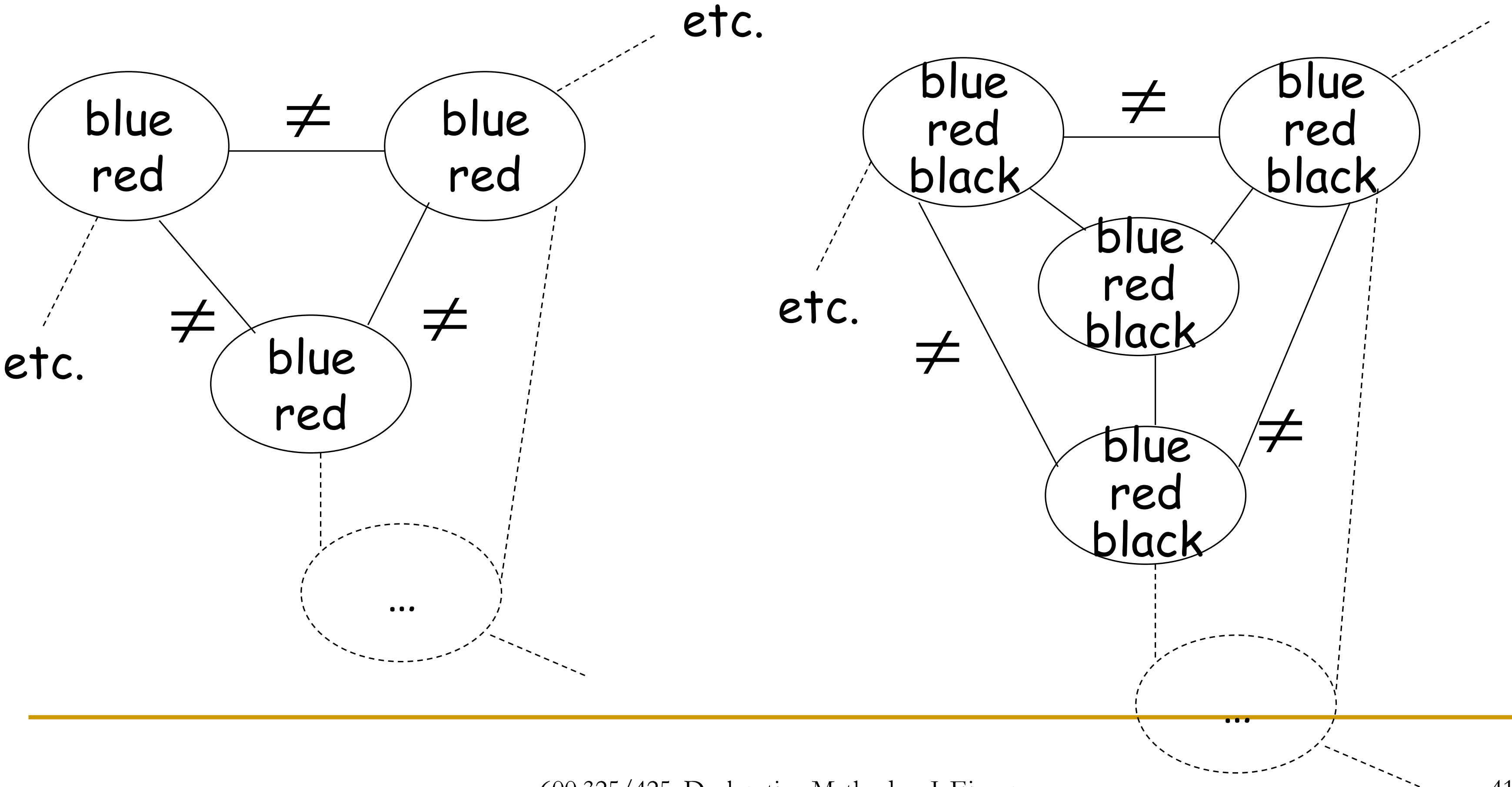
Moving on …

- We started with generalized arc consistency as our basic method.

- Bounds consistency is **weaker,** but often effective (and more efficient) for arithmetic constraints.
- What is **stronger** than arc consistency?

Looking at more than one constraint at a time

- What can you conclude here?
- When would you like to conclude it?
- Is generalized arc consistency enough? (1 constraint at a time)



Looking at more than one constraint at a time

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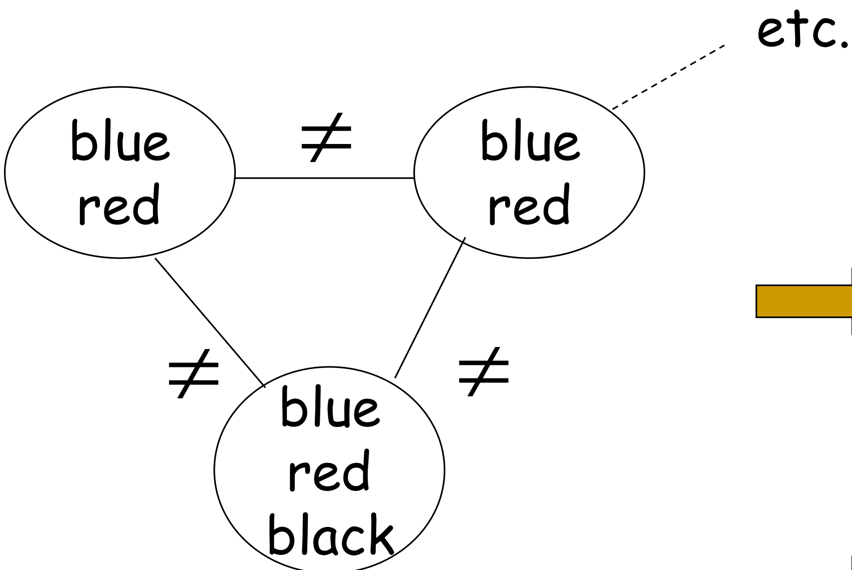
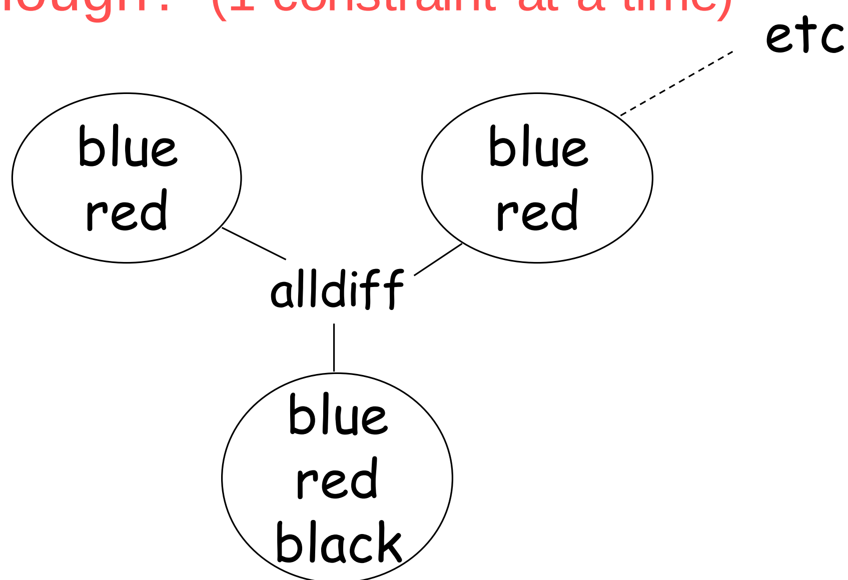


$X \neq Y$
$X \neq Z$
$Y \neq Z$

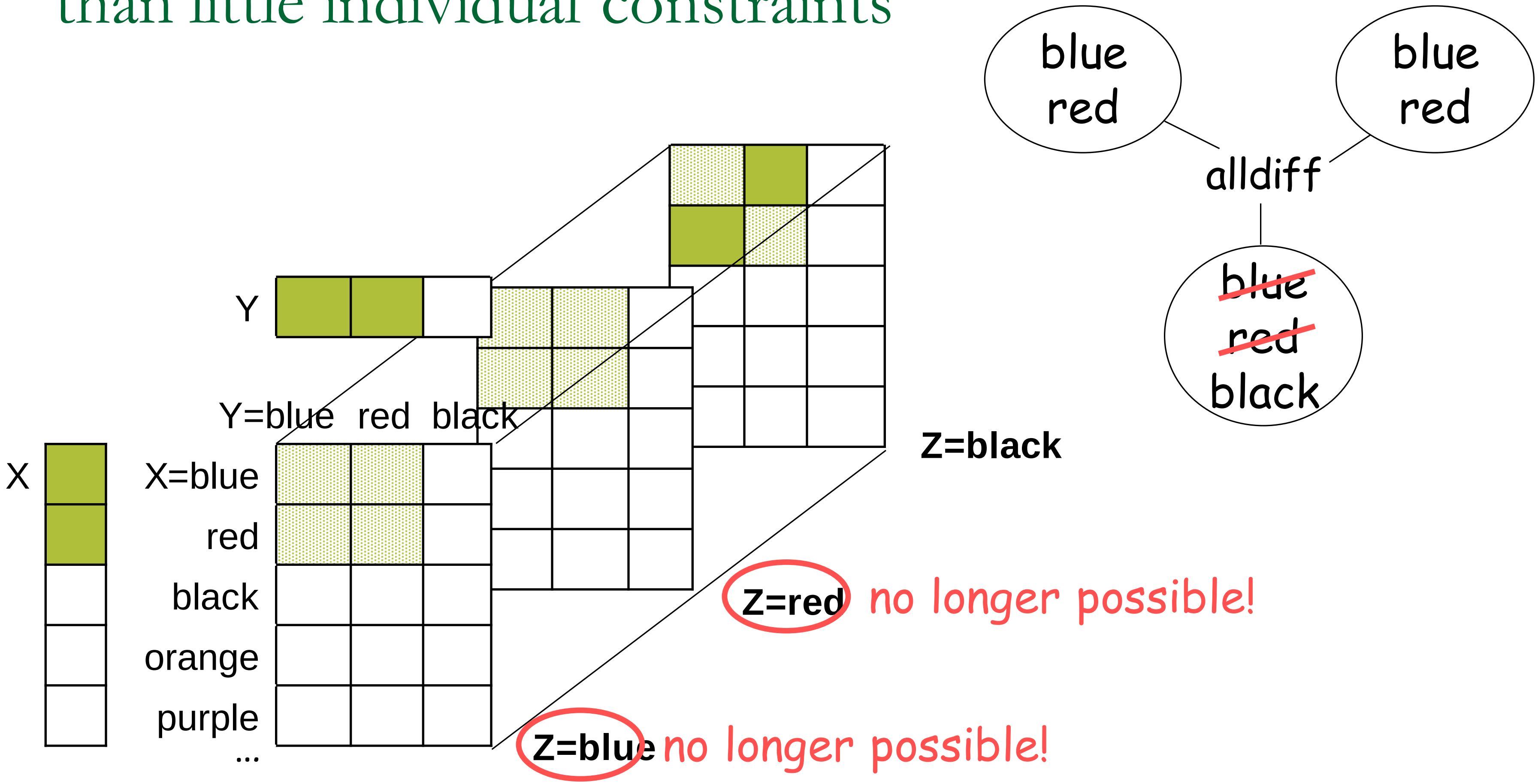


alldifferent(X,Y,Z)

"fuse" into bigger constraint
that relates more vars at once;
then do generalized arc consistency as usual.

What does that look like here?

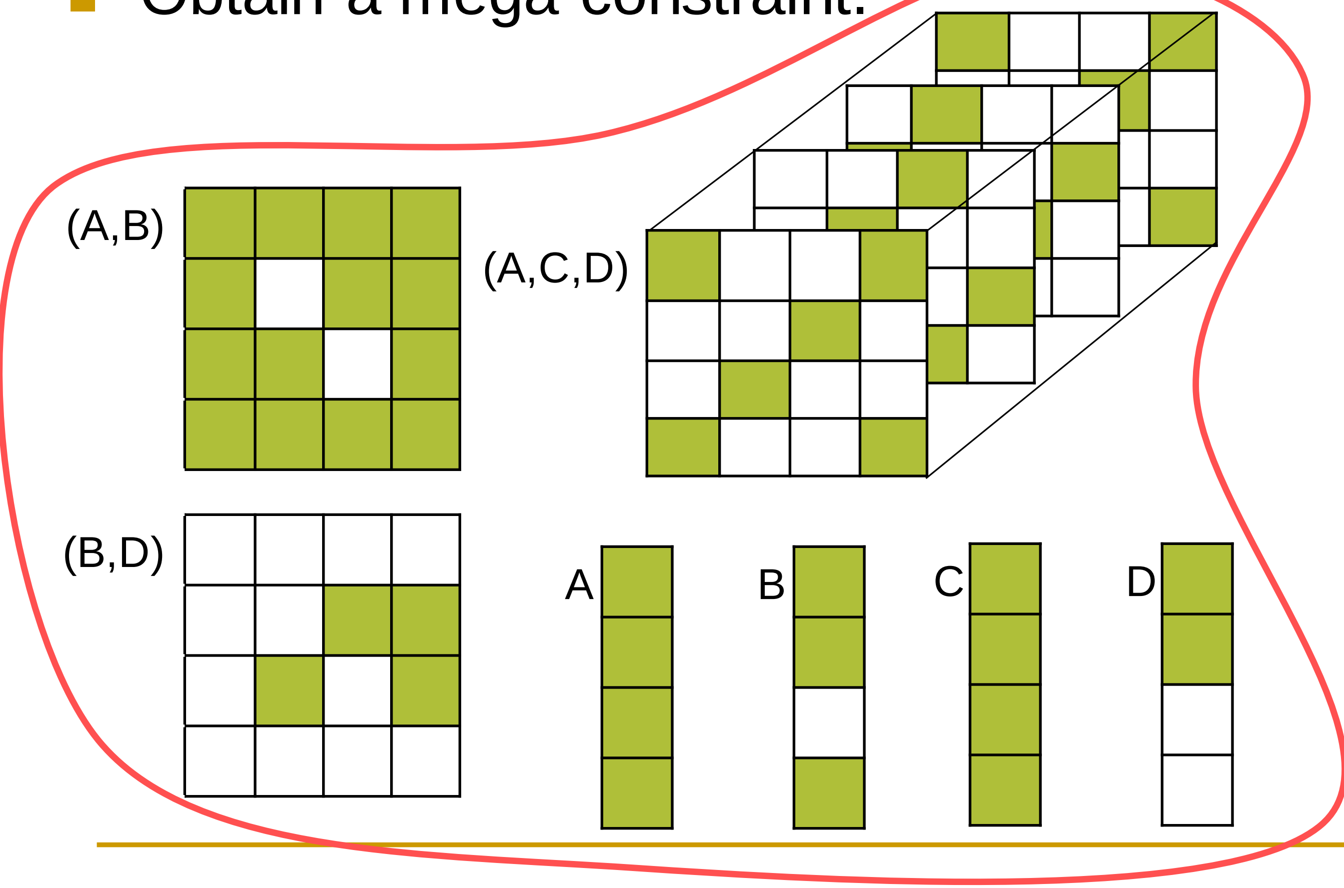
The big fused constraint has stronger effect than little individual constraints



alldifferent(X,Y,Z) **and** [X,Y]::[blue,red] **and** Z::[blue,red,black]

Joining constraints in general

- In general, can fuse several constraints on their common vars.
- Obtain a mega-constraint.



a <u>4-dimensional</u> grid showing possible values of the 4-tuple (A,B,C,D).

X≠Y, X≠Z, Y≠Z on the last slide happened to join into what we call alldifferent(X,Y,Z). But in general, this mega-constraint won't have a nice name. It's just a grid of possibilities.

Joining constraints in general

- This operation can be viewed (and implemented) as a natural **join** on databases.

a 4-column table listing possible values of the 4-tuple (A,B,C,D).

A	B
1	1
1	2
1	3
…	…

A	C	D
1	1	1
1	1	4
1	2	3
1	3	2
…	…	…

B	D
2	3
2	4
3	2
3	4

A
1
2
3
4

B
1
2
4

C
1
2
3
4

D
1
2

A	B	C	D
…	…	…	…

New mega-constraint.

How to use it?

- **project** it onto A axis (column) to get reduced domain for A
- if desired, **project** onto (A,B) plane (columns) to get new constraint on (A,B)

How many constraints should we join? Which ones?

- Joining constraints gives more powerful propagators.
- Maybe too powerful!: What if we join **all** the constraints?
  - We get a huge mega-constraint on all our variables.
  - How slow is it to propagate with this constraint (i.e., figure out the new variable domains)?
    - Boo! As hard as solving the whole problem, so NP-hard.
  - How does this interact with backtracking search?
    - Yay! "Backtrack-free."
    - We can find a first solution without any backtracking (if we propagate again after each decision).
      - Combination lock again.
    - Regardless of variable/value ordering.
- As always, try to find a good balance between propagation and backtracking search.

Options for joining constraints
(this slide uses the traditional terminology, if you care)

- Traditionally, all original constraints assumed to have ≤ 2 vars
  - 2-consistency or arc consistency: No joining (1 constraint at a time)
  - 3-consistency or path consistency:
    Join overlapping pairs of 2-var constraints into 3-var constraints

A note on binary constraint programs

- Traditionally, all original constraints assumed to have ≤ 2 vars

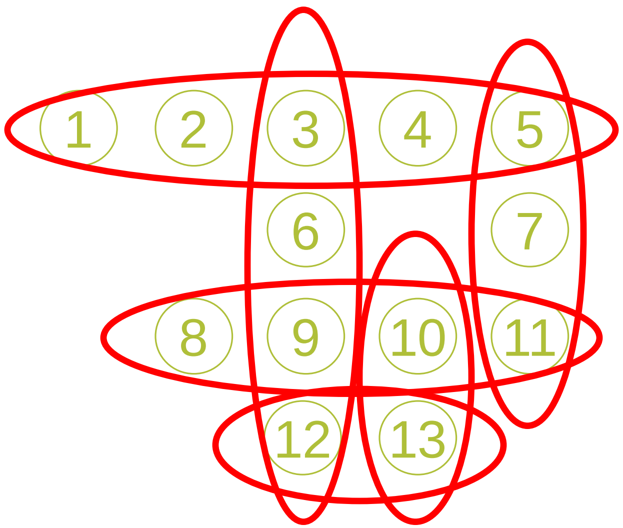
Tangential question: Why such a silly assumption?
Answer: Actually, it's completely general!
(just as 3-CNF-SAT is general: you can reduce any SAT problem to 3-CNF-SAT)

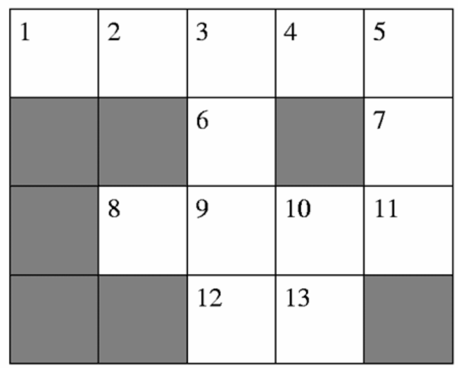
You can convert any constraint program to binary form. How?

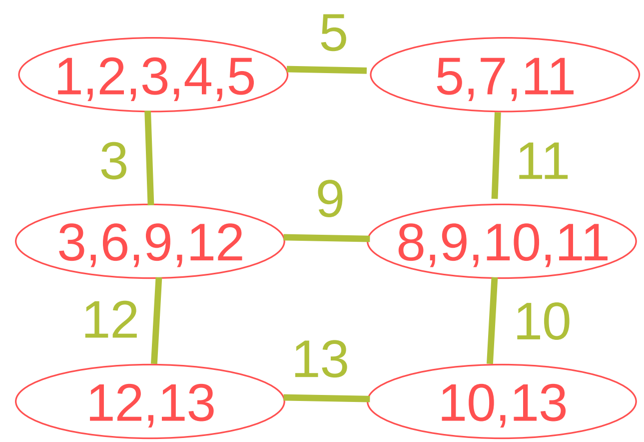
- Switching variables?
  - No: for SAT, that got us to ternary constraints (3-SAT, not 2-SAT).
  - But we're no longer limited to SAT: can go beyond boolean vars.
- If you have a 3-var constraint over A,B,C, replace it with a 1-var constraint … over a variable ABC whose values are triples!
- So why do we need 2-var constraints?
- To make sure that ABC's value agrees with BD's value in their B components. (Else easy to satisfy all the 1-var constraints!)

A note on binary constraint programs

- Traditionally, all original constraints assumed to have ≤ 2 vars
  - If you have a 3-var constraint over A,B,C, replace it with a 1-var constraint over a variable ABC whose values are triples
  - Use 2-var constraints to make sure that ABC's value agrees with BD's value in their B components







original ("primal") problem:
one variable per letter,
constraints over up to 5 vars

transformed ("dual") problem:
one var per word, 2-var constraints.
Old constraints → new vars!
Old vars → new constraints!

slide thanks to Rina Dechter (modified)

Options for joining constraints

(this slide uses the traditional terminology, if you care)

- Traditionally, all original constraints assumed to have ≤ 2 vars
  - 2-consistency or arc consistency: No joining (1 constraint at a time)
  - 3-consistency or path consistency:
    Join overlapping pairs of 2-var constraints into 3-var constraints
- More generally:
  - Generalized arc consistency: No joining (1 constraint at a time)
  - 2-consistency: Propagate only with 2-var constraints
  - 3-consistency: Join overlapping pairs of 2-var constraints into 3-var constraints, then propagate with **all** 3-var constraints
  - i-consistency: Join overlapping constraints as needed to get all mega-constraints of i variables, then propagate with those
  - strong i-consistency fixes a dumb loophole in i-consistency:
    Do 1-consistency, then 2-consistency, etc. up to i-consistency

Special cases of i-consistency propagation: When can you **afford** to join a lot of constraints?

- Suppose you have a lot of linear equations:
  - 3*X + 5*Y - 8*Z $= 0
  - -2*X + 6*Y - 2*Z $= 3
  - 6*X + 0*Y + 1*Z $= 8
- What does it mean to join these constraints?
  - Find values of X, Y, Z that satisfy all the equations simultaneously.
  - Hey! That's just ordinary math! Not exponentially hard.
  - Standard algorithm is $O(n^3)$: Gaussian elimination.
    - If system of eqns is overdetermined, will detect unsatisfiability.
    - If system of eqns is underdetermined, will not be able to finish solving, but will derive new, simpler constraints on the vars.

Special cases of i-consistency propagation: When can you **afford** to join a lot of constraints?

- Suppose you have a lot of linear inequalities:
  - 3*X + 5*Y - 8*Z #> 0
  - -2*X + 6*Y - 2*Z #> 3
  - 6*X + 0*Y + 1*Z #< 8
- What does it mean to join these constraints?
  - At least want something like bounds propagation: what are maximum and minimum values of X that are consistent with these constraints?
  - i.e., maximize X subject to the above inequality constraints
  - Again, math offers a standard algorithm! Simplex algorithm. (Polynomial-time in practice. Worst-case exponential, but there exist harder algorithms that are guaranteed polynomial.)
  - If algorithm says $X \leq 3.6$, we can conclude $X \leq 3$ since integer.

Why strong i-consistency is nice if you can afford it

- i-consistency: Join overlapping constraints as needed to get all mega-constraints of i variables, then propagate with those
- strong i-consistency fixes a dumb loophole in i-consistency: Do 1-consistency, then 2-consistency, etc. up to i-consistency

Thought experiment: At any time during backtracking search, we could arrange backtrack-freeness for the next 5 choices.

- Propagate to establish strong 5-consistency.
- If this leads to a contradiction, we're already UNSAT and must backtrack. Otherwise we can take 5 steps:
- Select next variable P, and pick any in-domain value for P.
  - Thanks to 5-consistency, this P value must be compatible with some tuple of values for (Q,R,S,T), the next 4 variables that we'll pick.
  - To help ourselves pick them, re-establish strong 4-consistency (possible because our original 5-consistency was strong). This narrows down the domains of (Q,R,S,T) given the decision for P.
- Now select next variable Q and an in-domain value for it.
  - And re-establish strong 3-consistency. Etc.

Trying 5-consistency here might result in UNSAT + backtracking

Why strong i-consistency is nice if you can afford it

- i-consistency: Join overlapping constraints as needed to get all mega-constraints of i variables, then propagate with those
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Trying 5-consistency here might result in UNSAT + backtracking.

But if we're lucky and our variable ordering has "induced width" < 5, we'll be able to re-establish strong 5-consistency after every decision. That will give us backtrack-free search for the entire problem!

(Easy to check in advance that a given var ordering has this property, but hard to tell whether any var ordering with this property exists.)

Variable elimination:

A good way to join lots of constraints, if that's what you want

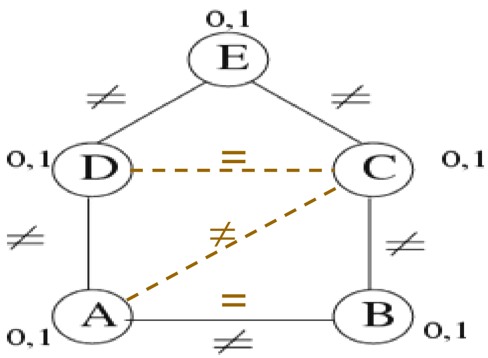
- If n = total number of variables, then propagating with strong n-consistency guarantees a **completely** backtrack-free search.
- In fact, even strong i-consistency guarantees this if the variable ordering has induced width < i. (We'll define this in a moment.)
- In fact, all we need is strong directional i-consistency. (Reduce P's domain enough to let us pick any (i-1) later vars in the ordering; by the time we get to P, we won't care anymore about picking earlier vars.)
- A more efficient variant of this is an "adaptive consistency" technique known as variable elimination.
  - "Adaptive" because we don't have to join constraints on all groups of i or fewer variables – only the groups needed to be backtrack-free.
  - Takes time $O(k^{(\text{induced width} + 1)})$, which could be exponential.
    - Some problems are considerably better than the worst case.
    - If the induced width turns out to be big, then approximate by joining fewer constraints than adaptive consistency tells you to.
      (Then your search might have to do some backtracking, after all.)

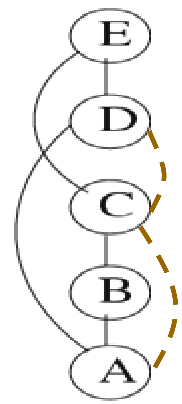
Variable elimination

- Basic idea: Suppose we have variables A, B, … Y, Z.
  - Join **all** the constraints that **mention** Z, and project Z out of the resulting new mega-constraint.
  - So the new mega-constraint allows any combination of values for A, … Y that is fully consistent with at least one value of Z.
    - Note: It mentions only variables that were co-constrained with Z.
  - If we choose A, B, … Y during search so as to be consistent with **all** constraints, including the mega-constraint, then the mega-constraint guarantees that there is some consistent way to choose Z as well.
- So now we have a “smaller” problem involving only constraints on A, B, … Y.
  - So repeat the process: join **all** the constraints that **mention** Y …
- When we’re all done, our search will be backtrack free, if we are careful to use the variable ordering A, B, … Z.

Variable elimination

- Each variable keeps a "bucket" of all the constraints that mention it (and aren't already in any higher bucket).





Bucket E: $E \neq D$, $E \neq C$

Bucket D: $D \neq A$ → $D = C$

Bucket C: $C \neq B$ → $A \neq C$

Bucket B: $B \neq A$ → $B = A$

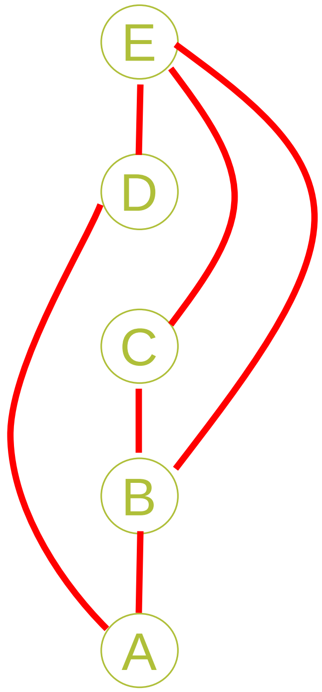
Bucket A: → contradiction

join all constraints in E's bucket
yielding a new constraint on D (and C)
now join all constraints in D's bucket …

slide thanks to Rina Dechter

Variable ordering matters for variable elimination!

Variable ordering A,B,C,D,E.
Draw an edge between two variables if some constraint mentions both of them.



Eliminate E first.
E interacted with B,C,D.
O(k^4) time and space to join all constraints on E and construct a new mega-constraint relating B,C,D.
(Must enumerate all legal B,C,D,E tuples ("join"), to find the legal B,C,D tuples ("project").)

example thanks to Rina Dechter

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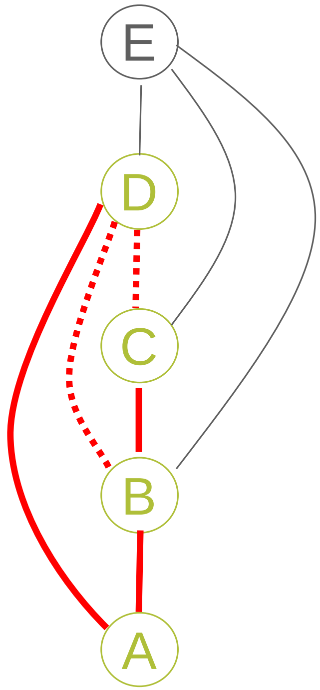

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Alas, this new constraint adds new graph edges!
D now interacts with B and C, not just A. ☹
Next we eliminate D: O(k^4) time and space to construct a new mega-constraint relating A,B,C.

example thanks to Rina Dechter

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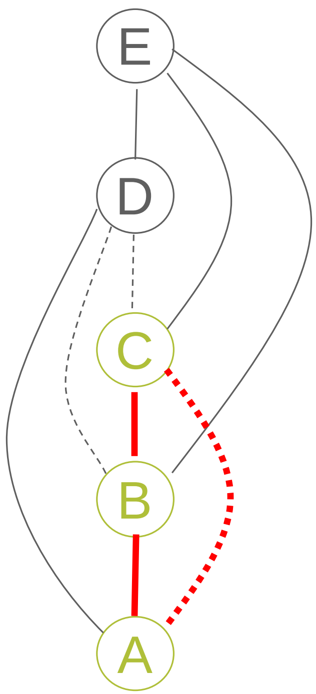

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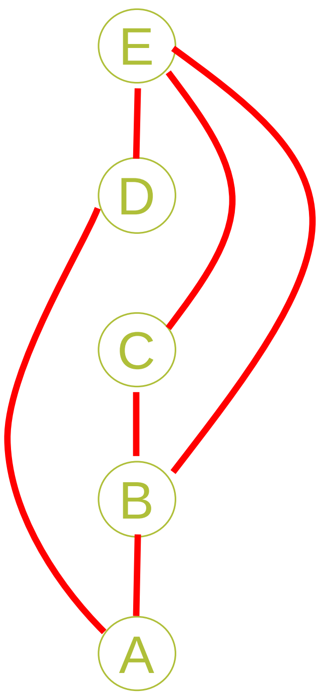
example thanks to Rina Dechter

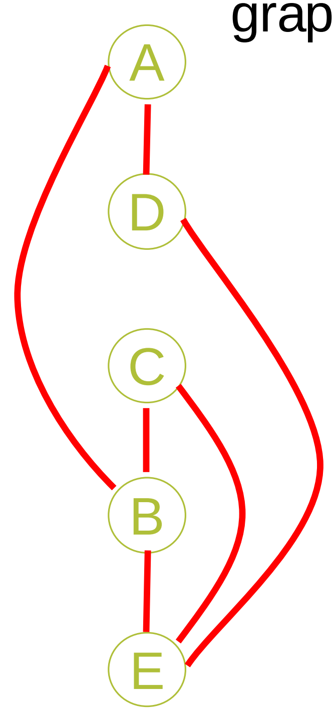
Variable ordering matters for variable elimination!

Variable ordering A,B,C,D,E.
Draw an edge between two variables if some constraint mentions both of them.

This better variable ordering takes only $O(k^3)$ time at each step.
By the time we eliminate any variable, it has at most 2 edges, not 3 as before.
We say the "induced width" of the graph along this ordering is 2.



$O(k^4)$ time to eliminate E; likewise D.



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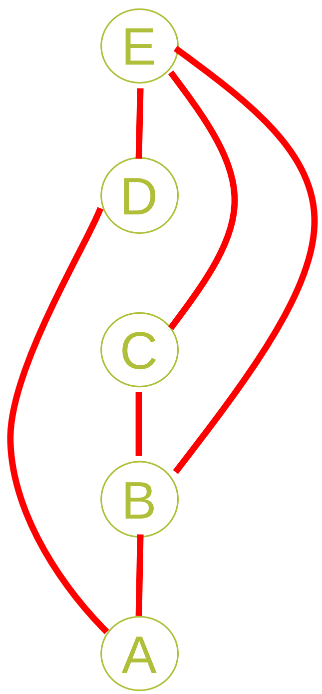

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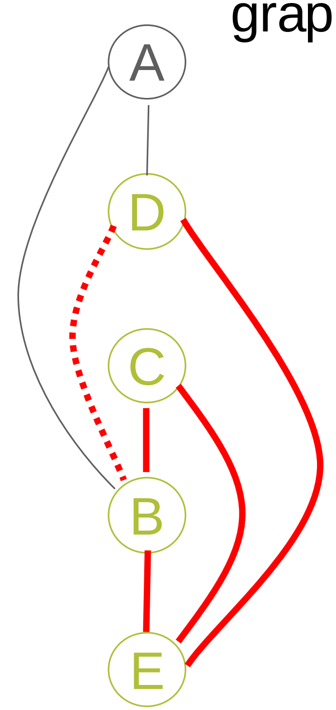

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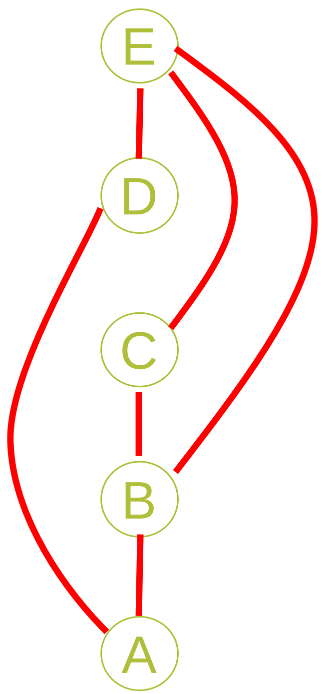

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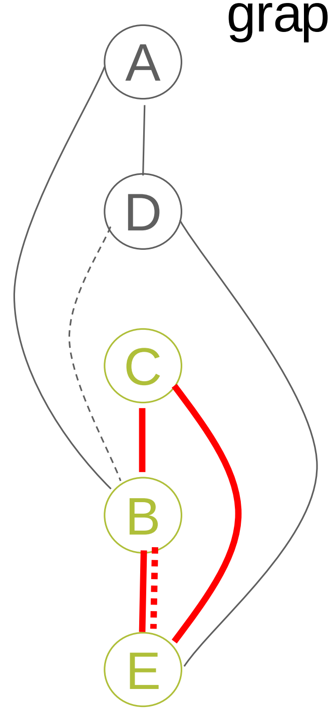

New mega-constraint on B and E, but they were **already** connected

example thanks to Rina Dechter

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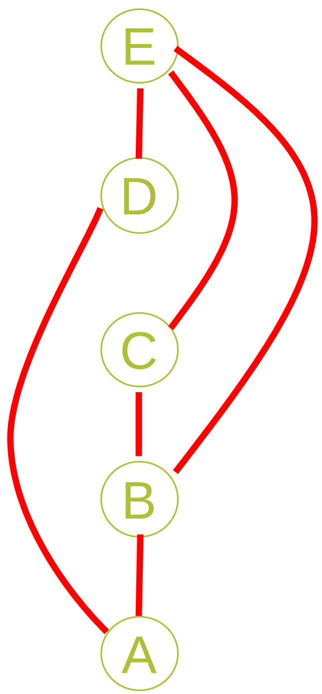

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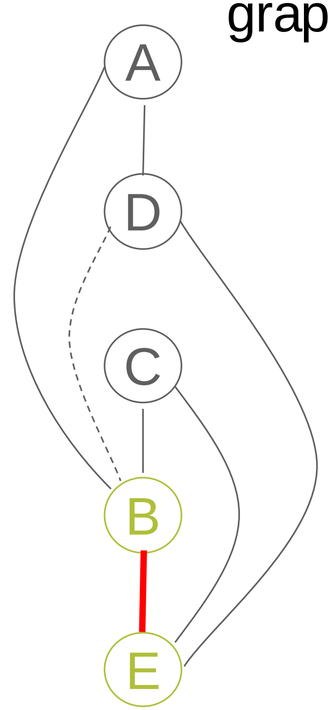

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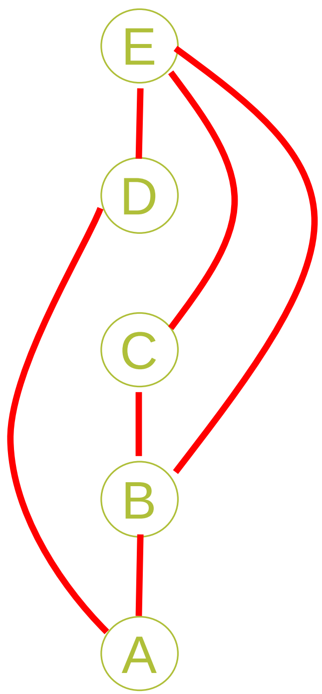

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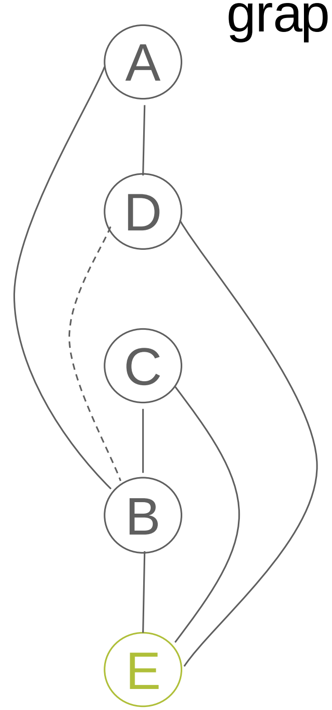


Probably want to use a var ordering that has minimum induced width.
But even determining the minimum induced width (the "elimination width" or "treewidth") is NP-complete.
In practice, can use a greedy heuristic to pick the var ordering.

example thanks to Rina Dechter

Gaussian elimination is just variable elimination!

3*X + 5*Y - 8*Z #= 0
-2*X + 6*Y - 2*Z #= 3
6*X + 0*Y + 1*Z #= 8

Eliminate variable Z by joining equations that mention Z

Add 8*equation 3 to equation 1

3*X + 5*Y - 8*Z #= 0
8(6*X + 0*Y + 1*Z) #= 64
51*X + 5*Y #= 64

Add 2*equation 3 to equation 2

-2*X + 6*Y - 2*Z #= 3
2(6*X + 0*Y + 1*Z) #= 16
10*X + 6*Y #= 19

51*X + 5*Y #= 64
10*X + 6*Y #= 19

Next, eliminate variable Y by adding (-5/6)*equation 2 to equation 1

…

Davis-Putnam is just variable elimination!

Remember from 2 weeks ago …

- Function DP(φ): // φ is a CNF formula
  - **if** φ has no clauses, return SAT
  - **else if** φ contains an empty clause, return UNSAT
  - **else**
    - pick any variable Z that still appears in φ
    - **return** DP($\underbrace{(\varphi \wedge Z) \vee (\varphi \wedge \sim Z)}$)

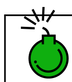
we eliminate this variable

by resolution

We put this argument into CNF before recursing
This procedure (resolution) eliminates all copies of Z and ~Z.
Fuses each pair (V v W v ~Z) ^ (X v Y v Z) into (V v W v X v Y)
The collection of resulting clauses is our "mega-constraint."
May square the number of clauses ☹

"Minesweeper" CSP

Which squares have a bomb? Squares with numbers don't. Other squares might. Numbers tell how many of the eight adjacent squares have bombs. We want to find out if a given square can possibly have a bomb….

	1	1			
1	1	1			
0	0	1			
1	1	2			

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

“Minesweeper” CSP

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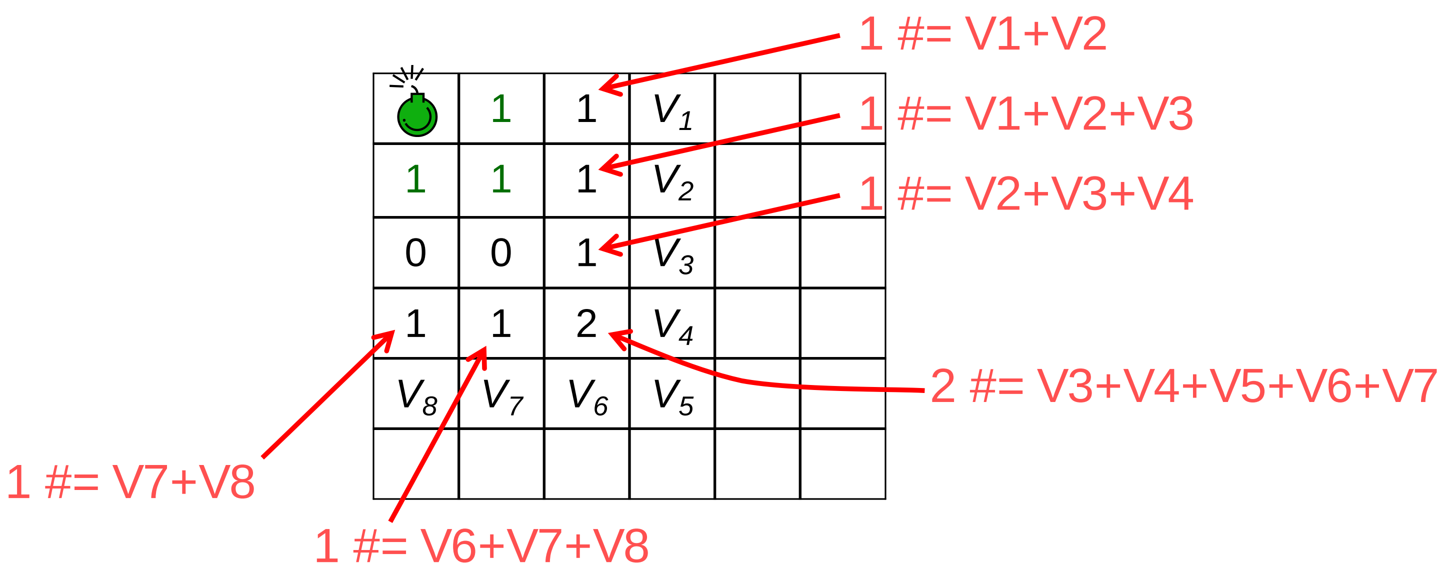


[V1, V2, V3, V4, V5, V6, V7, V8] :: 0..1, % number of bombs in that square

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

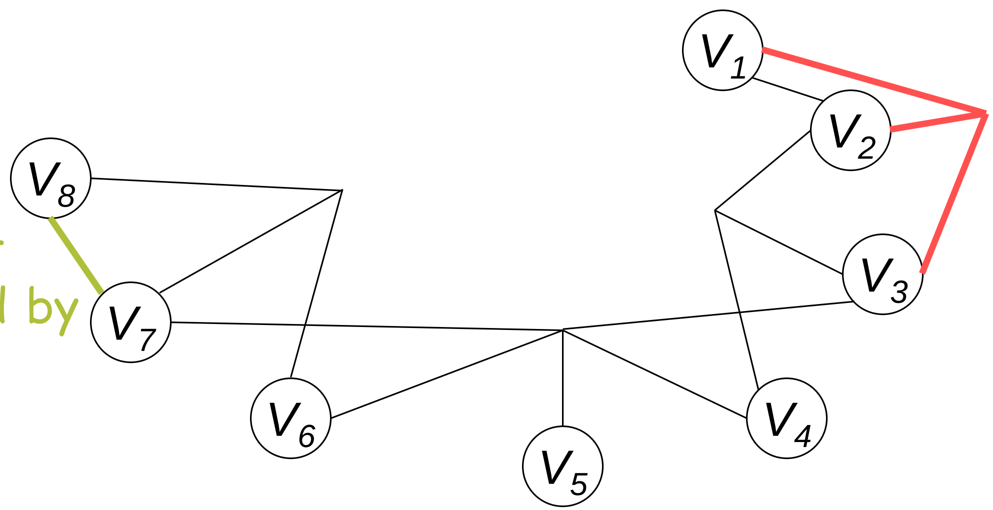
"Minesweeper" CSP

	1	1	V_1		
1	1	1	V_2		
0	0	1	V_3		
1	1	2	V_4		
V_8	V_7	V_6	V_5		

[V1, V2, V3, V4, V5, V6, V7, V8] :: 0..1, % number of bombs in that square

1 #= V1+V2, 1 #= V1+V2+V3,
1 #= V2+V3+V4, 2 #= V3+V4+V5+V6+V7, 1 #= V6+V7+V8,
1 #= V7+V8

edge shows that V7,V8 are linked by a 2-variable constraint



hyperedge shows that V1,V2,V3 are linked by a 3-variable constraint

slide thanks to Andrew Moore (modified): http://www.cs.cmu.edu/~awm/tutorials

“Minesweeper” CSP

	1	1	V_1		
1	1	1	V_2		
0	0	1	V_3		
1	1	2	V_4		
V_8	V_7	V_6	V_5		

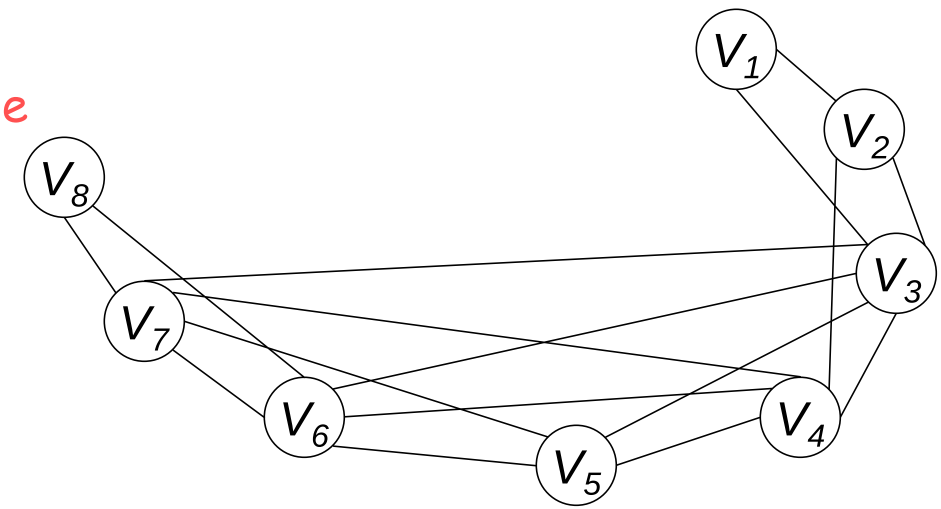
What would you guess about best variable ordering, e.g., for variable elimination?

A minesweeper graph has a natural “sequence.” A good order will act like dynamic programming and let us process different parts of graph more or less independently.

[V1, V2, V3, V4, V5, V6, V7, V8] :: 0..1, % number of bom

1 #= V1+V2, 1 #= V1+V2+V3, 1 #= V2+V3+V4,
2 #= V3+V4+V5+V6+V7, 1 #= V6+V7+V8, 1 #= V7+V8

change style of graph to what we used before: link two vars if they appear together in any constraint



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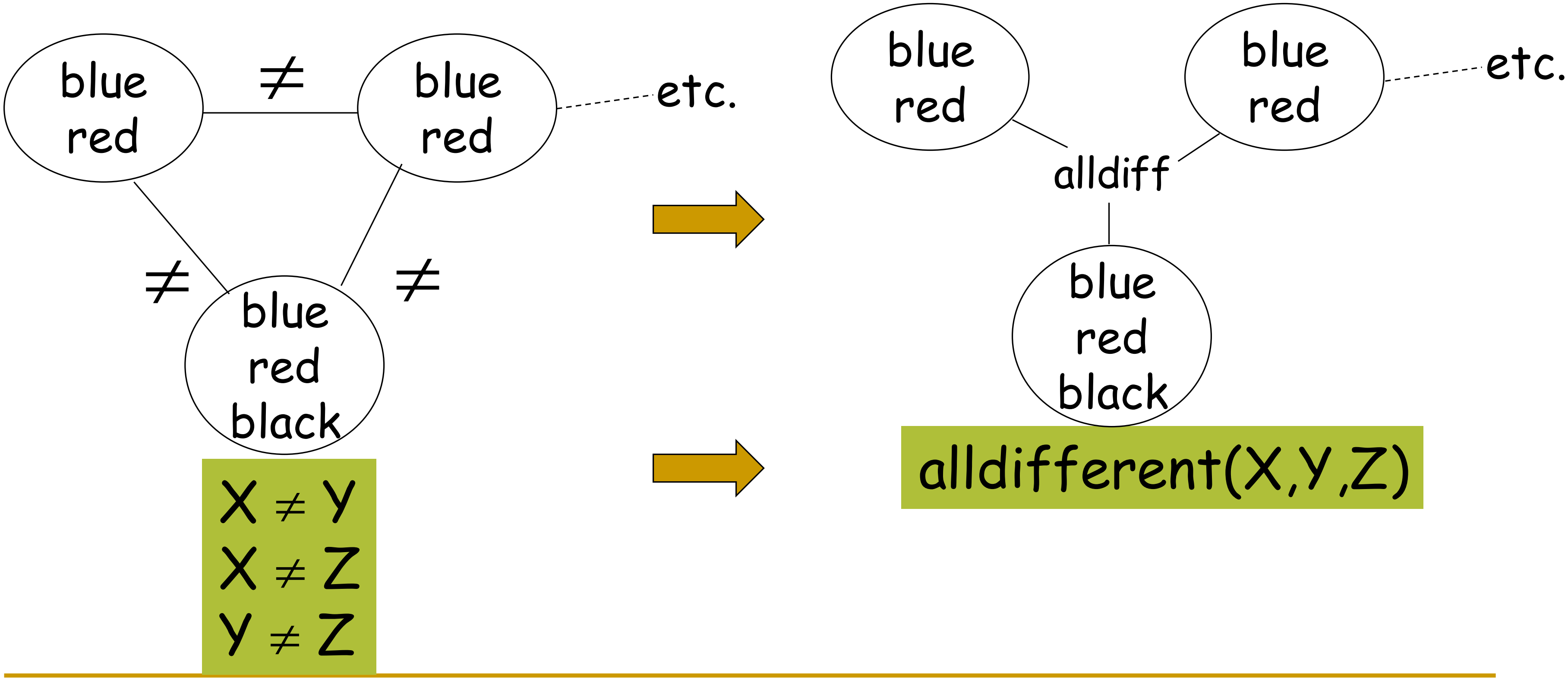
A case study of propagators: Propagators for the alldifferent constraint

- Earlier, we joined many ≠ into one alldifferent constraint.
- But how can we **efficiently** propagate alldifferent?



A case study:
Propagators for the alldifferent constraint

- Earlier, we joined many ≠ into one alldifferent constraint.
- But how can we **efficiently** propagate alldifferent?
- Often it's useful to write alldifferent(a whole bunch of vars).

- Option 1:
  Treat it like a collection of pairwise ≠

  So if we learn that X=3, eliminate 3 from domains of Y,Z,…

  No propagation if we learn that X::[3,4].
  Must narrow X down to a **single** value in order to propagate.

A case study: Propagators for the alldifferent constraint

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- Option 2:
  Just like option 1 (a collection of pairwise ≠),
  but add the "pigeonhole principle."

  That is, do a quick check for unsatisfiability:
  for alldifferent(A,B,…J) over 10 variables, be sure to fail if the union of their domains becomes smaller than 10 values.
  That failure will force backtracking.

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- Option 3:
  Generalized arc consistency as we saw before.

Example: scheduling workshop speakers at different hours.

A::3..6, B::3..4, C::2..5, D::3..4, alldifferent([A,B,C,D])

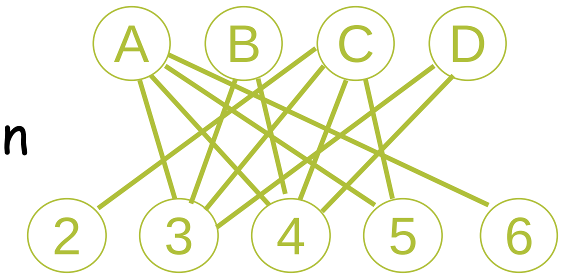
Note that B, D "use up" 3 and 4 between them.

So A, C can't be 3 or 4.

We deduce A::5..6 and C::[2,5].

A case study:
Propagators for the alldifferent constraint

A bipartite graph showing the domain constraints.



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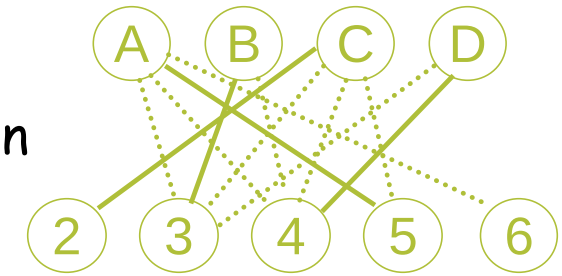

An assignment to [A,B,C,D] that also satisfies alldiff is a **matching** of size 4 in this graph.
(a term from graph theory)

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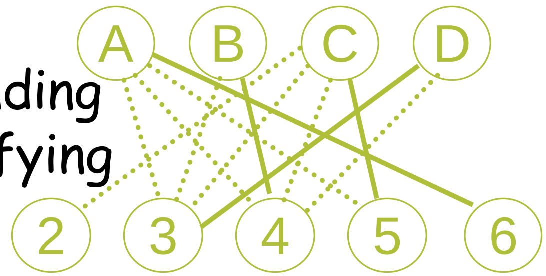
Propagators for the alldifferent constraint

Here's a different matching, corresponding to a different satisfying assignment.



To reduce domains, we need to detect edges that are not used in **any** full matching.

Clever algorithm does this in time sqrt(n)*m, where n = num of nodes and m = num of edges.

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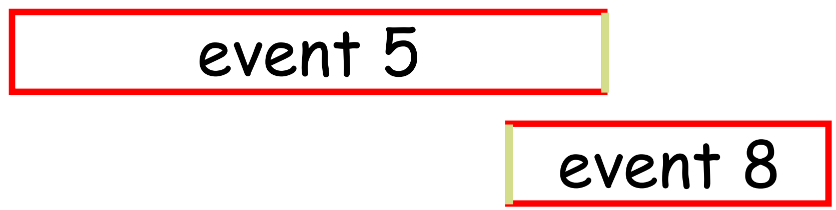
Another case study:
"Edge-finding" propagators for scheduling

- Want to schedule a bunch of talks in the same room, or a bunch of print jobs on the same laser printer.
- Use special scheduling constraints (and others as well).



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- So if we learn that start8 < end5, we can conclude …

Another case study:
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- So if we learn that start8 < end5, we can conclude …
that end8 ≤ start5 (i.e., event 8 is *completely* before event 5).

One more idea: Relaxation

(a third major technique, alongside propagation and search)

- Suppose you have a huge collection of constraints – maybe exponentially many – too many to use all at once.
- Ignore some of them, giving a "relaxed" problem. Solve that first:
  - If you were lucky, solution satisfies **most** of the ignored constraints too.
  - Add in any few of the constraints that were violated and try again. The new constraints "cut off" the solution you just found.

- That's how traveling salesperson problems are solved!
  - http://www.tsp.gatech.edu/methods/dfj - clear explanation
  - http://www.tsp.gatech.edu/methods/cpapp - interactive Java applet
- Common to relax constraints saying that some vars must be integers:
  - Then you can use traditional fast equation solvers for real numbers.
  - If you get fractional solutions, add new linear constraints ("cutting planes") to cut those off.
  - In particular, integer linear programming (ILP) is NP-complete – and many problems can naturally be reduced to ILP and solved by an ILP solver.

Branch and bound
(spiritually related to relaxation)

- Constraint satisfaction problems:
  - find **one** satisfying assignment
  - find **all** satisfying assignments
    - just continue with backtracking search
  - find **best** satisfying assignment
    - i.e., minimize Cost, where Cost #= Cost1 + Cost2 + …
      - Where would this be practically useful?
      - Use the "minimize" predicate in ECLiPSe (see assignment)
      - Useful ECLiPSe syntax:
Cost #= (A #< B) + 3*(C #= D) + …
where A #< B is "bad" and counts as a cost of 1 if it's true, else 0
    - How? Could find all assignments and keep a running minimum of Cost. Is there a better way?

Branch and bound

(spiritually related to relaxation)

- find **best** satisfying assignment
  - i.e., minimize Cost, where Cost #= Cost1 + 3*Cost2 + …
  - How? Could find **all** assignments by backtracking, and pick the one with minimum Cost. Is there a better way?
  - Yes! Suppose the **first** assignment we find has Cost=72.
  - So add a new constraint Cost #< 72 before continuing with backtracking search.
  - The new constraint "cuts off" solutions that are the same or worse than the one we already found.
  - Thanks to bounds propagation, we may be able to figure out that Cost ≥ 72 while we're still high up in the search tree. Then we can cut off a whole branch of search.
    - (Similar to A* search, but the heuristic is automatically computed for you by constraint propagation!)

Branch and bound example

- Want to minimize Cost
- Cost #= V1 + V2 + V3 + …
- How will bounds propagation help cut off solutions?

- Assignment problem: Give each person the job that makes her happiest
  - How to formalize happiness?
  - What are the constraints?
    - How to set up alldifferent to avoid conflicts?
  - How will branch and bound work?

Branch and Bound

Example: assignment problem

Let us consider n people who need to be assigned n jobs, one person per job (each person is assigned exactly one job and each job is assigned to exactly one person).

Suppose that the cost of assigning job j to person i is C(i,j).

Find an assignment with a minimal total cost.

Mathematical description.

Find $(s_1,\ldots,s_n)$ with s_i in $\{1,\ldots n\}$ denoting the job assigned to person i such that:

- $s_i <> s_k$ for all i<>k (different persons have to execute different jobs)
- $C(1,s_1)+C(2,s_2)+\ldots.+C(n,s_n)$ is minimal

Branch and Bound

Example:

$$C = \begin{matrix} 9 & 2 & 7 \\ 6 & 4 & 3 \\ 5 & 8 & 1 \end{matrix}$$

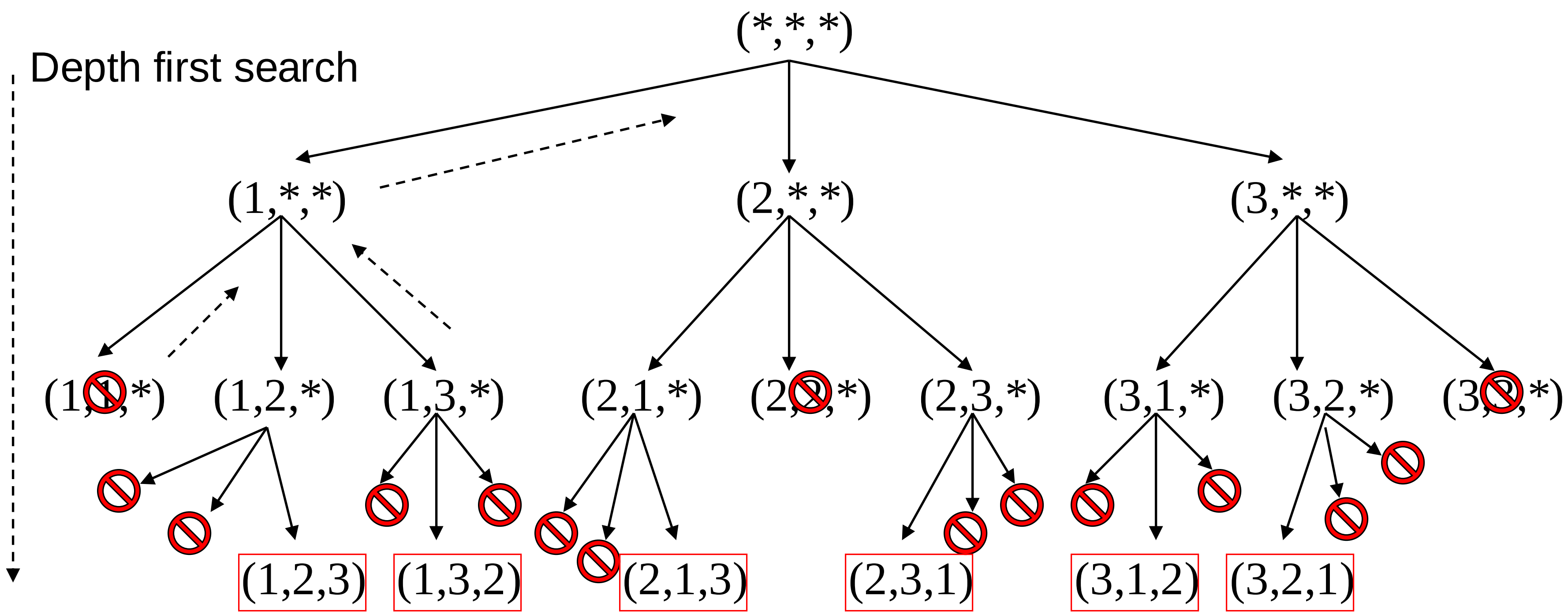
Idea for computing a lower bound for the optimal cost:

the cost of any solution will be at least the sum of the minimal values on each row (initially 2+3+1=6). This lower bound is not necessarily attained because it could correspond to a non-feasible solution ((2,3,1) doesn't satisfy the constraint)

This is exactly what bounds propagation will compute as a lower bound on Cost!

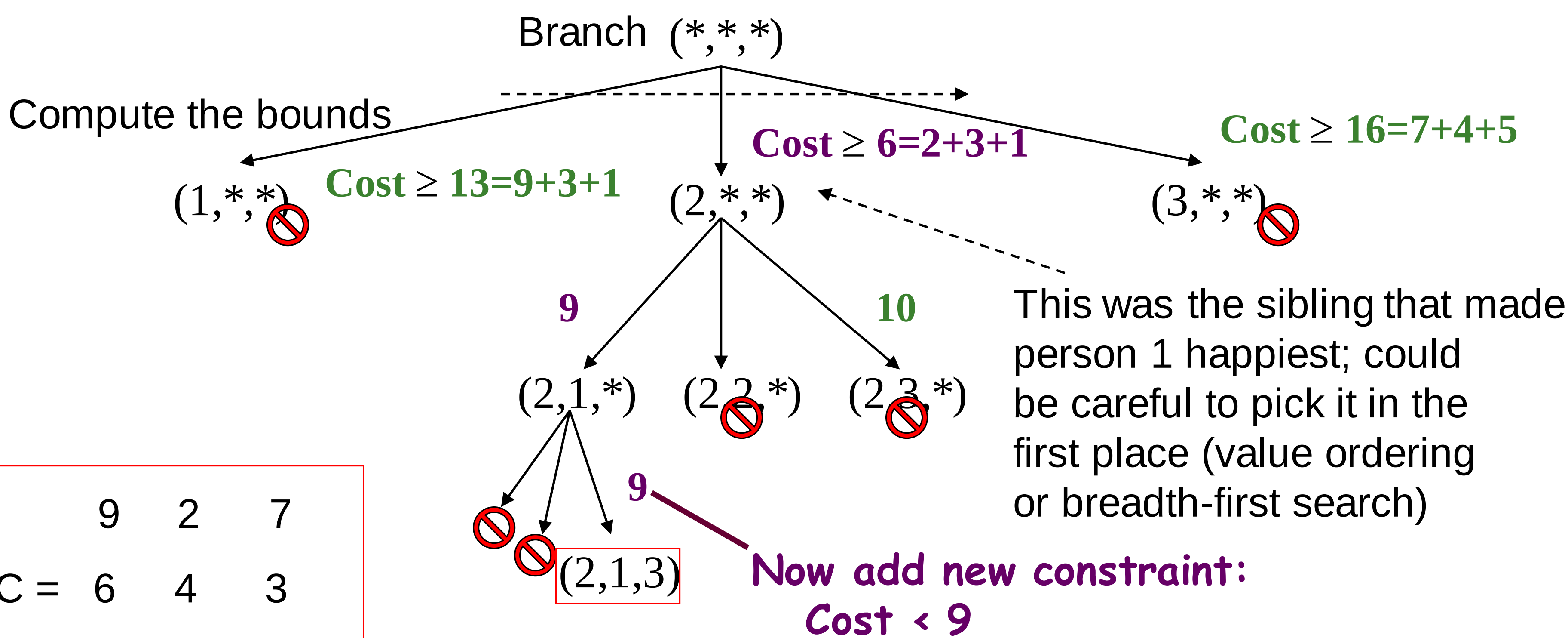
Branch and bound

State space tree for permutations generation (classical backtracking)



Branch and bound

State space tree for optimal assignment (use lower bounds to establish the feasibility of a node)



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