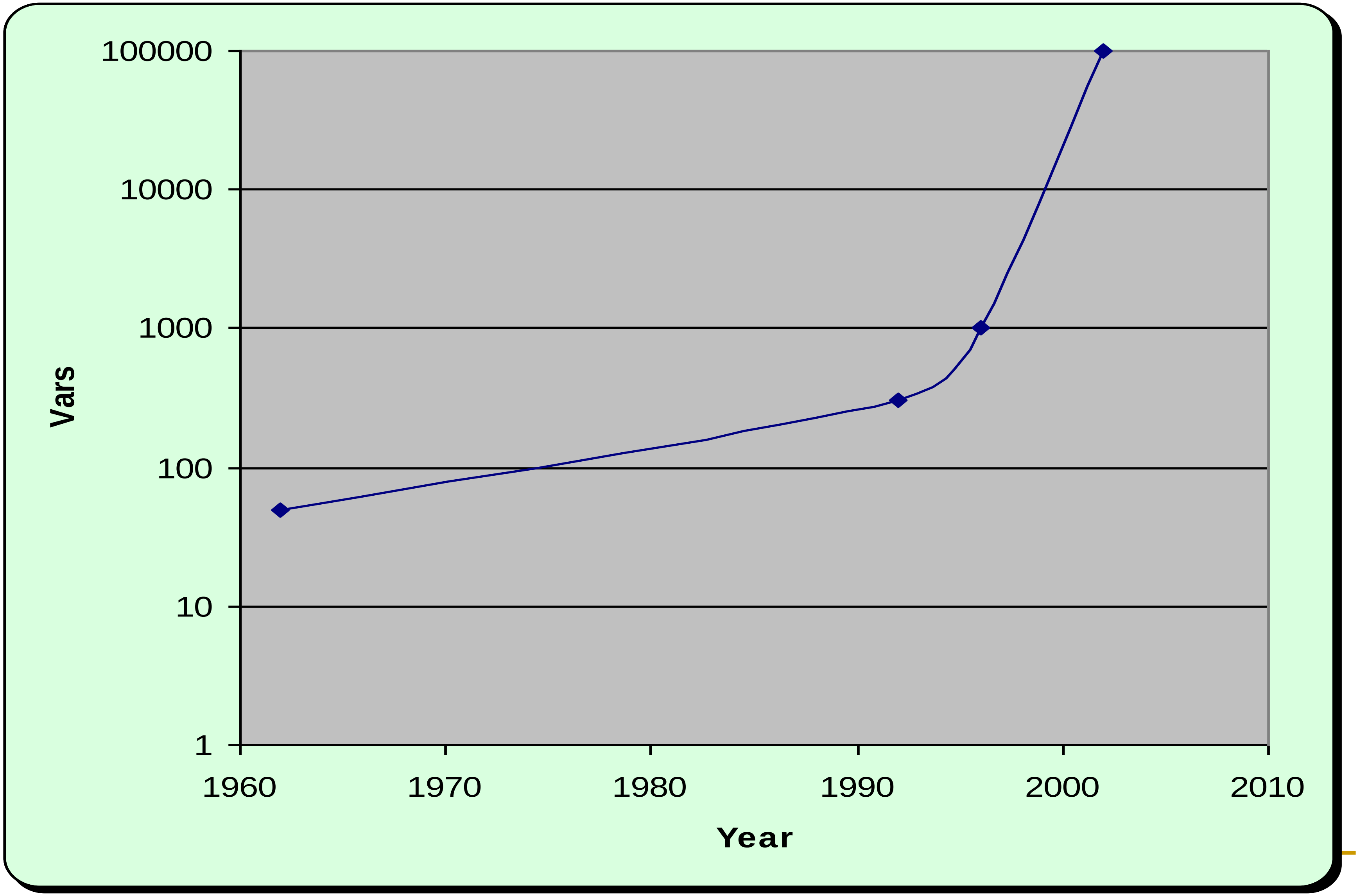
Satisfiability Solvers

Part 1: Systematic Solvers

SAT solving has made some progress…



slide thanks to Daniel Kroening

SAT solving has made some progress…

"Whoa! until 1990, SAT speedups didn't even keep pace with hardware speedups."

Why is this wrong & misleading?

graphs thanks to Daniel Kroening and thocp.net

Exhaustive search

// Suppose formula uses 5 variables: A, B, C, D, E

- for A $\in$ {false, true}
  - for B $\in$ {false, true}
    - for C $\in$ {false, true}
      - for D $\in$ {false, true}
        - for E $\in$ {false, true}

          if formula is true

            immediately return (A,B,C,D,E)
- return UNSAT

Exhaustive search

// Suppose formula uses 5 variables: A, B, C, D, E

- for A $\in$ {0, 1}
  - for B $\in$ {0, 1}
    - for C $\in$ {0, 1}
      - for D $\in$ {0, 1}
        - for E $\in$ {0, 1}

          if formula is true

          immediately return (A,B,C,D,E)
- return UNSAT

Short-circuit evaluation

- for A ∈ {0, 1}
  - If formula is now definitely true, immediately return SAT
  - If formula is now definitely false, loop back & try next value of A
  - for B ∈ {0, 1}
    - If formula is now def. true, immediately return SAT
    - If formula is now def. false, loop back & try next value of B
    - for C ∈ {0, 1}
      - …
      - for D ∈ {0, 1}
        - …
        - for E ∈ {0, 1}

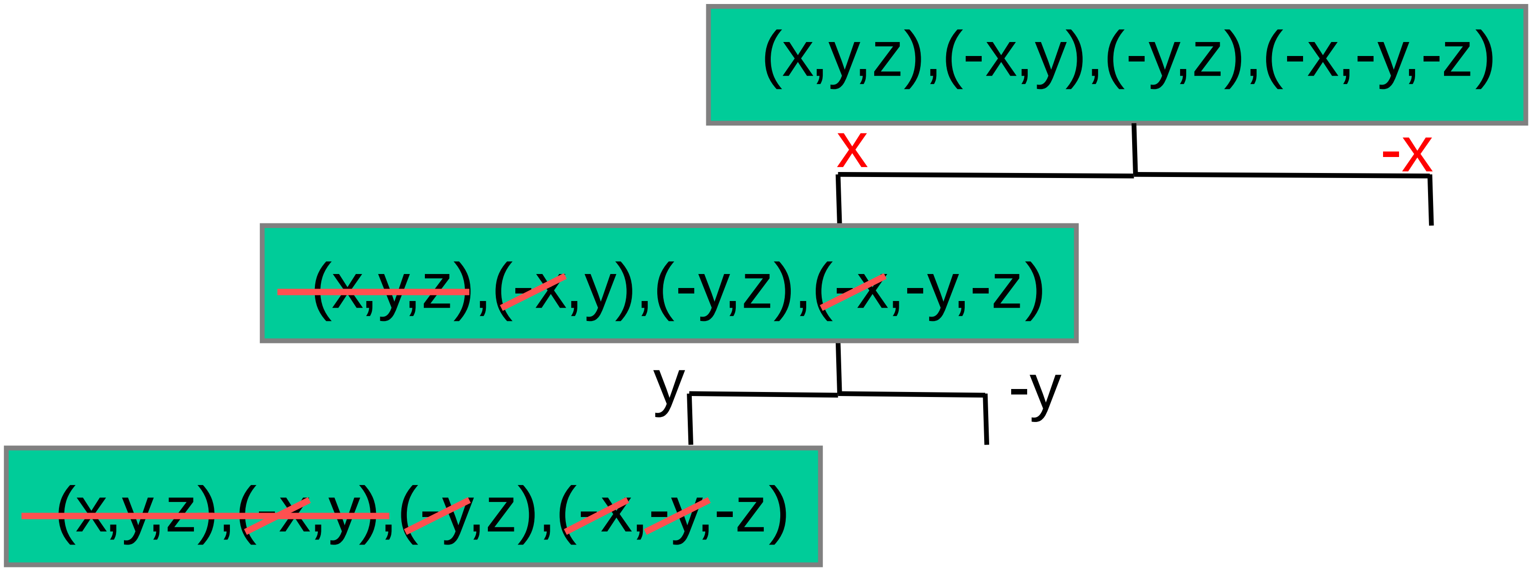
          if formula is true

          immediately return SAT
- return UNSAT

How would we tell? When would these cases happen?

Short-circuit evaluation

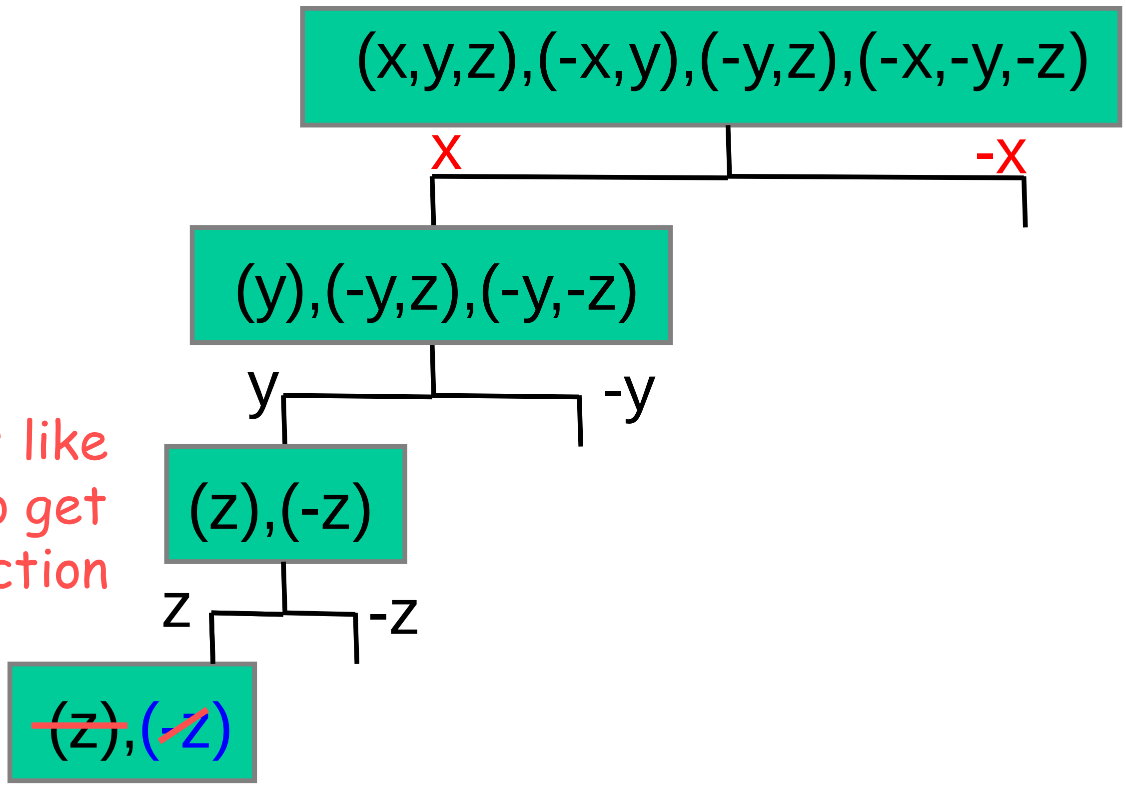
(X v Y v Z) ^ (~X v Y) ^ (~Y v Z) ^ (~X v ~Y v ~Z)



slide thanks to Daniel Kroening (modified)

Short-circuit evaluation

(X v Y v Z) ^ (~X v Y) ^ (~Y v Z) ^ (~X v ~Y v ~Z)

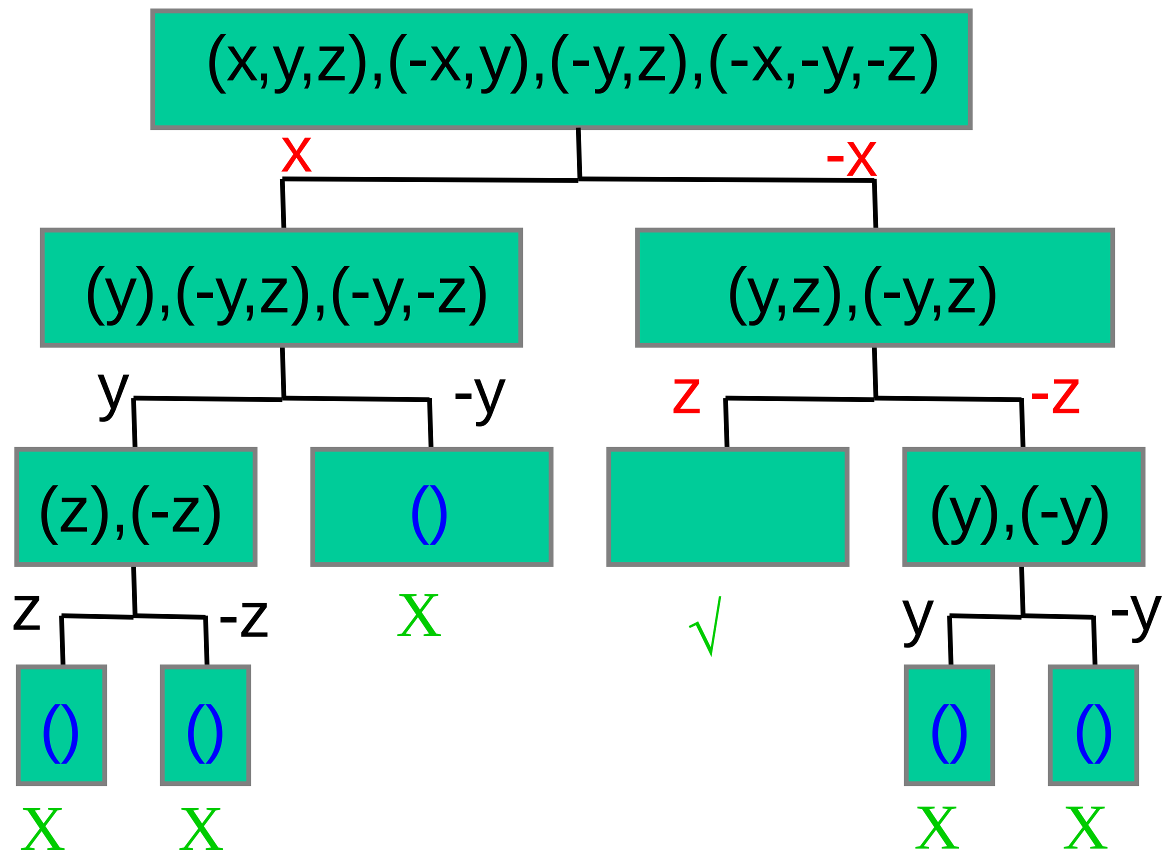


Hmm, looks like we're about to get a contradiction

slide thanks to Daniel Kroening (modified)

Short-circuit evaluation

(X v Y v Z) ^ (~X v Y) ^ (~Y v Z) ^ (~X v ~Y v ~Z)



slide thanks to Daniel Kroening (modified)

Variable ordering might matter

- How do we pick the order A,B,C,D,E? And the order 0,1?
  - Any order is correct, but affects how quickly we can short-circuit.
- Suppose we have A v D among our clauses:
  - **Trying A=0** forces D=1
  - So after setting A=0, it would be best to consider D next
    - (Rule out D=0 **once**, not separately for all vals of (B,C))
- What if we **also** have ~A v B v C?
  - **Trying A=1** forces B=1 or C=1
  - So after setting A=1, it might be good to consider B or C next
- So, can we order {B,C,D,E} differently for A=0 and A=1?
  - What did we actually do on previous slide?

Variable and value ordering is an important topic.
Hope to pick a satisfying assignment on the first try!
We'll come back to this … many heuristics.

The most important variable ordering trick

"Unit propagation" or "Boolean constraint propagation"

- Suppose we try **A=0** …
- Then all clauses containing ~A are satisfied and can be deleted.
- We must also remove A from all clauses.
- Suppose one of those clauses is (A v D). It becomes (D), a "unit clause."
- Now we know **D=1**. Might as well deal with that right away.
- Chain reaction:
  - All clauses containing D are satisfied and can be deleted.
  - We must also remove ~D from all clauses
  - Suppose we also have (~D v C) … ? It becomes (C), a "unit clause."
  - Now we know **C=1**. Might as well deal with that right away.
    - Suppose we also have (A v ~C v ~B) …? A is already gone…

The most important variable ordering trick

"Unit propagation" or "Boolean constraint propagation"

- This leads to a "propagation" technique:
  - If we have any unit clause (1 literal only), it is a **fact** that lets us immediately shorten or eliminate other clauses.
  - What if we have **more than one** unit clause?
    - Deal with all of them, in any order
  - What if we have **no** unit clauses?
    - Can't propagate facts any further.
    - We have to guess: pick an unassigned variable X and try both X=0, X=1 to make more progress.
    - This never happens on the LSAT exam.
- For satisfiable instances of 2-CNF-SAT, this finds a solution in O(n) time with the right variable ordering (which can also be found in O(n) time)!

Constraint propagation tries to eliminate future options <u>as soon as possible</u>, to avoid eliminating them repeatedly later.

We'll see this idea again!

DLL algorithm (often called DPLL)

Davis-(Putnam)-Logemann-Loveland

Why start ***after*** picking A or ~A? Maybe original formula already had some unit clauses to work with …

- for each of A, ~A
  - Add it to the formula and try doing unit propagation
  - If formula is now def. true, immediately return SAT
  - If formula is now def. false, abandon this iteration (loop back)
  - for each of B, ~B
    - Add it to the formula and try doing unit propagation
    - If formula is now def. true, immediately return SAT
    - If formula is now def. false, abandon this iteration (loop back)
    - for each of C, ~C
      - …
      - for e
        - …
        - for each of E, ~E

          …
- return UNSAT

don't wanna hardcode all these nested loops

What if we want to choose variable ordering as we go?

What if propagating A already forced B=0? Then we have **already** established ~B and propagated it. So skip this.

DLL algorithm (often called DPLL)

Cleaned-up version

- Function DLL(φ): // φ is a CNF formula
  - **while** φ contains at least one unit clause: // unit propagation
    - pick any unit clause X // we **know** X's value
    - remove all clauses containing X // so substitute that val,
    - remove ~X from all remaining clauses // eliminating var X!
  - **if** φ now has no clauses left, return SAT
  - **else if** φ now contains an empty clause, return UNSAT
  - **else** // the hard case
    - pick any variable Z that still appears in φ // choice affects speed
    - **if** DLL(φ ^ Z) = SAT **or** DLL(φ ^ ~Z) = SAT // try both if we must
    - **then** return SAT
    - **else** return UNSAT
- *How would you fix this to actually return a satisfying assignment?*
- *Can we avoid the need to copy φ on a recursive call?*

Hint: assume the recursive call does so

Compare with the older DP algorithm

Davis-Putnam

- **DLL(φ) recurses twice (unless we're lucky and the first recursion succeeds):**
  - **if** DLL(φ ^ X) = SAT **or** DLL(φ ^ ~X) = SAT // for some X we picked
  - Adds unit clause X or ~X, to be simplified out by unit propagation (along with **all** copies of X **and** ~X) as soon as we recursively call DLL .
- **DP(φ) tail-recurses once, by incorporating the "or" into the formula:**
  - **if** DP((φ ^ X) v (φ ^ ~X)) = SAT // for some X we picked
  - No branching: we tail-recurse <u>once</u> … on a formula with n-1 variables!
    - Done in n steps. We'll see this "**variable elimination**" idea again…
  - So what goes wrong?
    - We have to put the argument into CNF first.
    - This procedure (resolution) eliminates all copies of X **and** ~X. Let's see how …

Compare with the older DP algorithm

Davis-Putnam

some two clauses in φ

Resolution fuses each pair (V v W v ~X) ^ (X v Y v Z) into (V v W v Y v Z)
Justification #1: Valid way to eliminate X (reverses CNF → 3-CNF idea).
Justification #2: Want to recurse on a CNF version of $((\varphi \wedge X) \vee (\varphi \wedge \sim X))$
Suppose $\varphi = \alpha \wedge \beta \wedge \gamma$
where α is clauses with ~X, β with X, γ with neither
Then $((\varphi \wedge X) \vee (\varphi \wedge \sim X)) = (\alpha' \wedge \gamma) \vee (\beta' \wedge \gamma)$ by unit propagation
where α' is α with the ~X's removed, β' similarly.
$= (\alpha' \vee \beta') \wedge \gamma = (\alpha'_1 \vee \beta'_1) \wedge (\alpha'_1 \vee \beta'_2) \wedge \ldots \wedge (\alpha'_{99} \vee \beta'_{99}) \wedge \gamma$

- **if** DP((φ ^ X) v (φ ^ ~X)) = SAT // for some X we picked
- No branching: we tail-recurse once … on a formula with n-1 variables!
- But we have to put the argument into CNF first.
- This procedure (resolution) eliminates all copies of X **and** ~X.
  - Done in n steps. So what goes wrong?
  - Size of formula can square at each step.
  - (Also, not so obvious how to recover the satisfying assignment.)

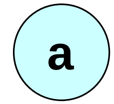
Basic DLL Procedure

$(a' + b + c)$
$(a + c + d)$
$(a + c + d')$
$(a + c' + d)$
$(a + c' + d')$
$(b' + c' + d)$
$(a' + b + c')$
$(a' + b' + c)$

slide thanks to Sharad Malik (modified)

Basic DLL Procedure

(a’ + b + c)
(a + c + d)
(a + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)

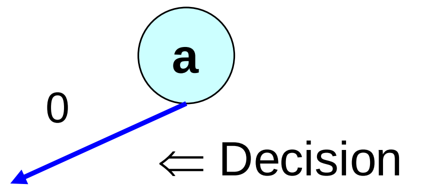


slide thanks to Sharad Malik (modified)

Basic DLL Procedure

Green means "crossed out"

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)



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Basic DLL Procedure

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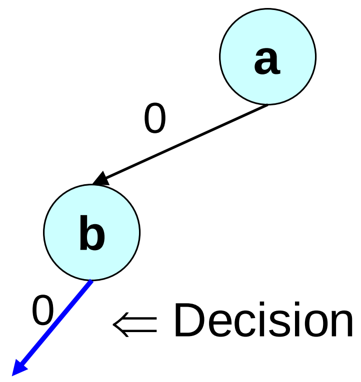

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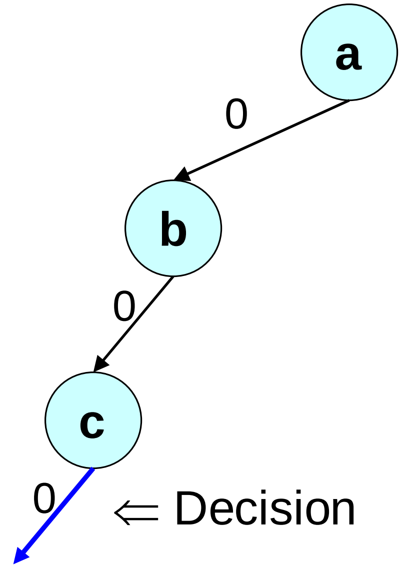

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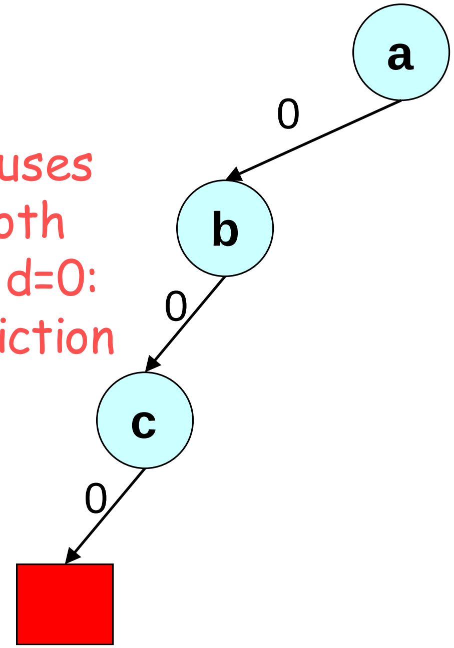
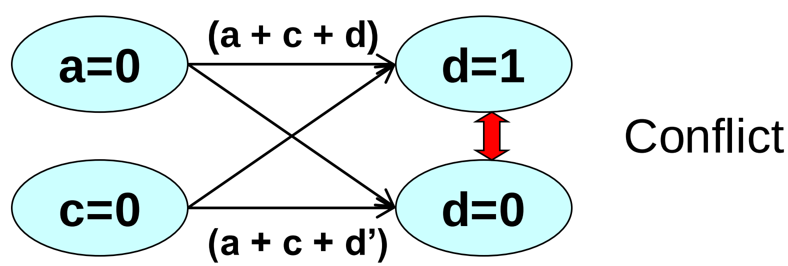
Unit clauses force both d=1 and d=0: contradiction



Implication Graph
(shows that the problem was caused by a=0 ^ c=0; nothing to do with b)



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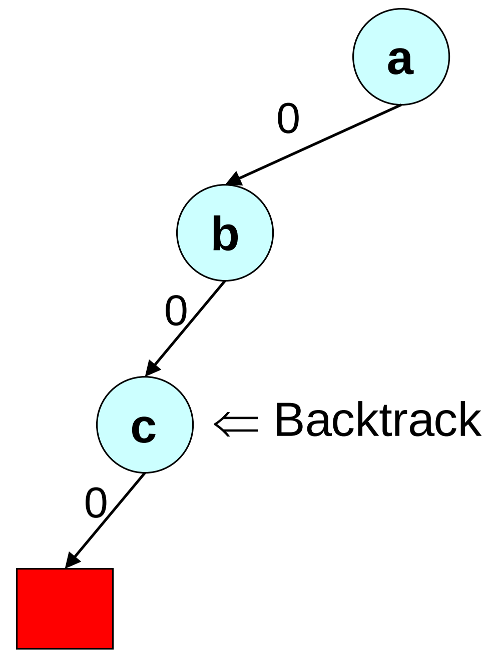

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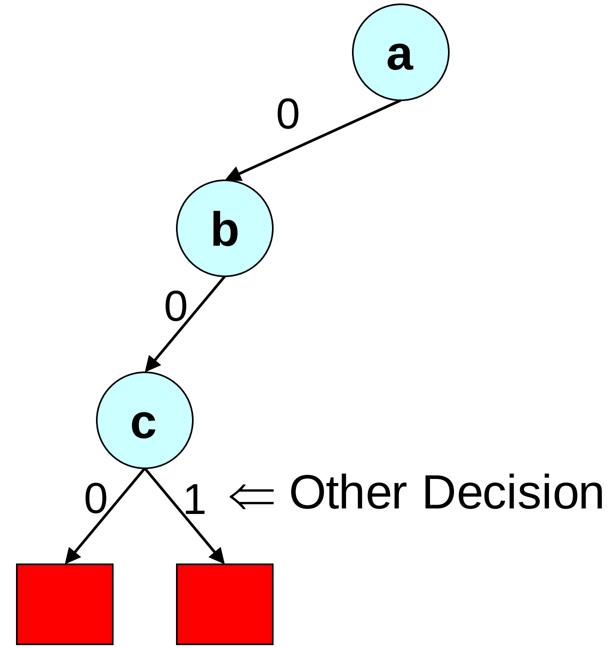

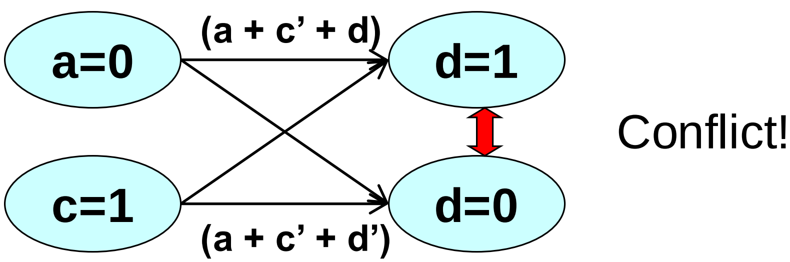

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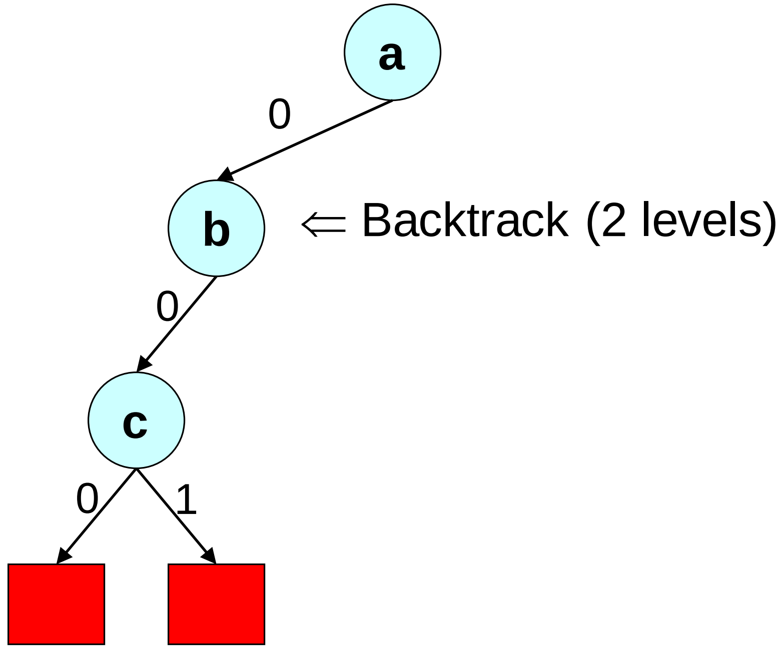

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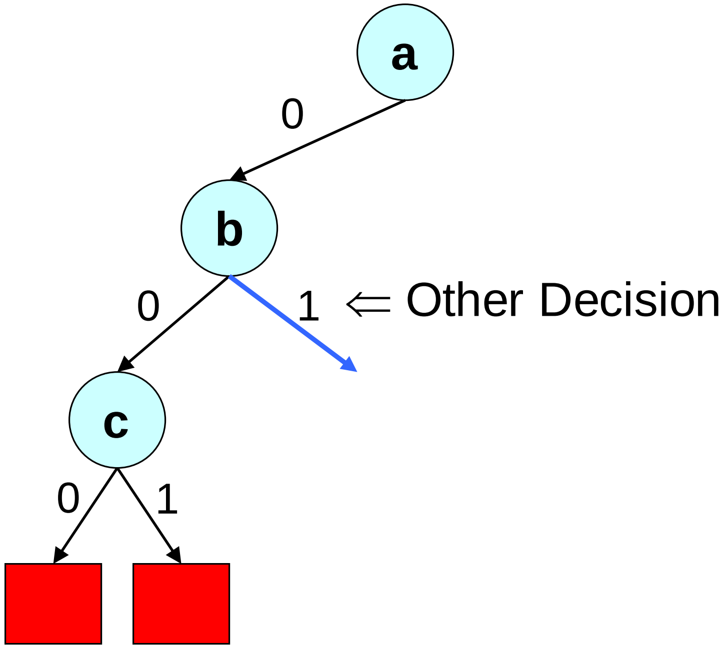

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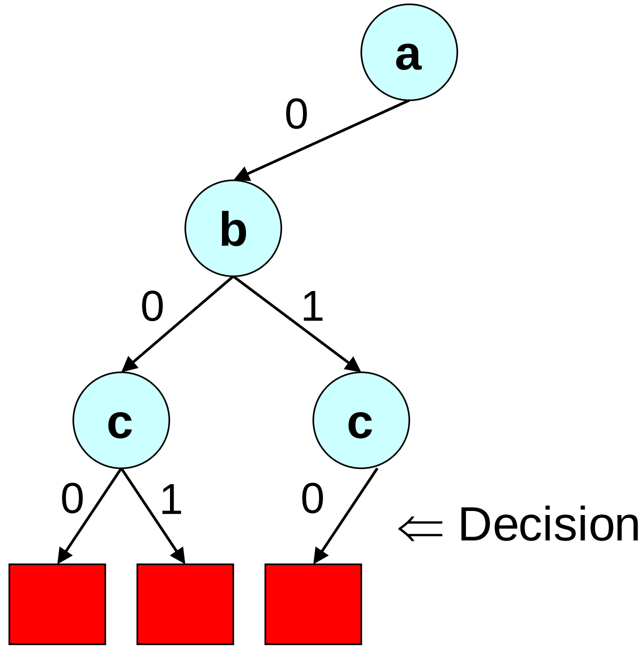

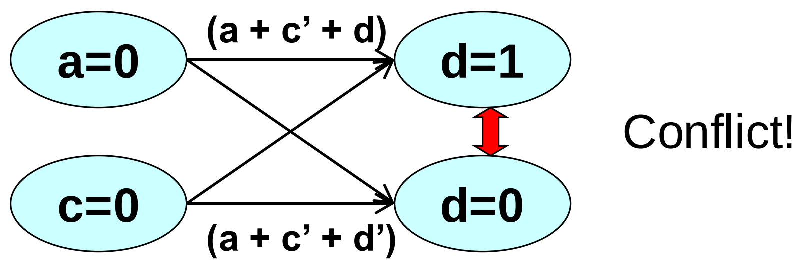

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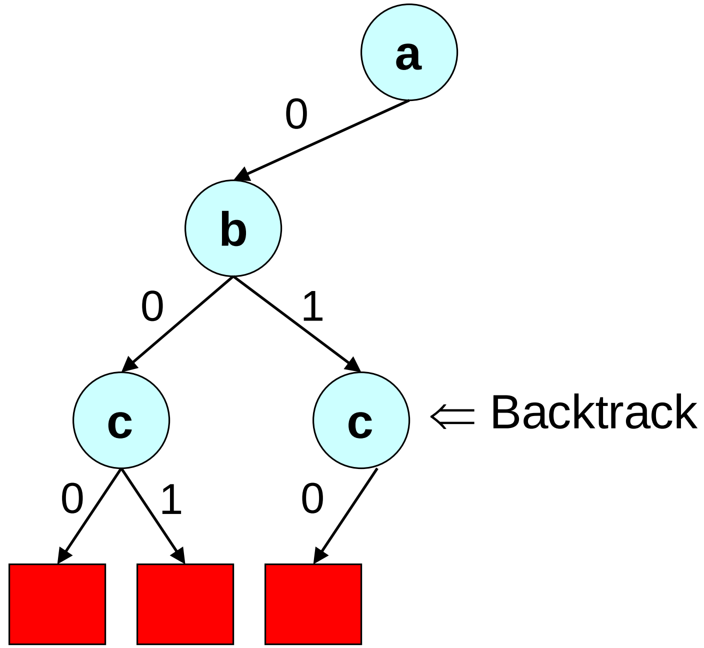

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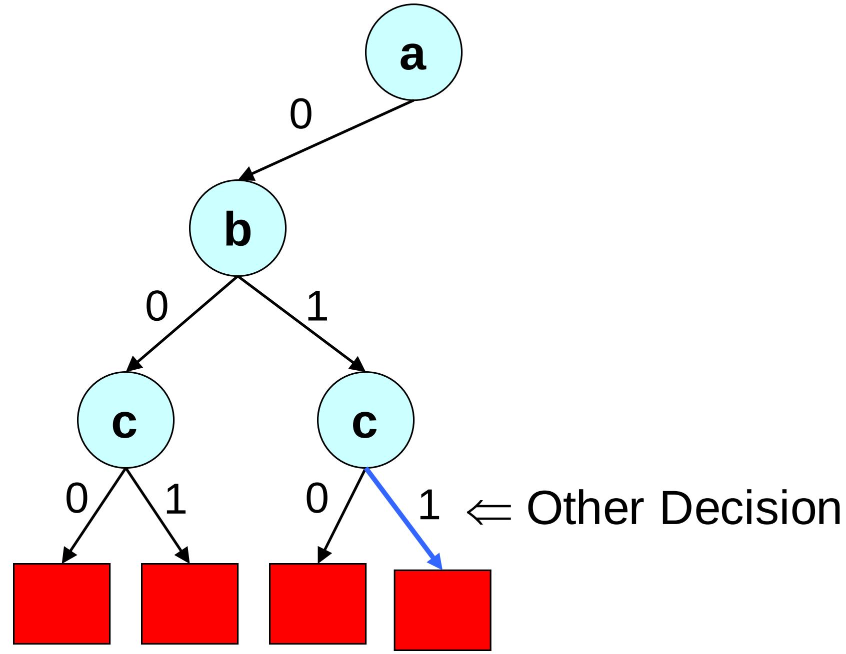

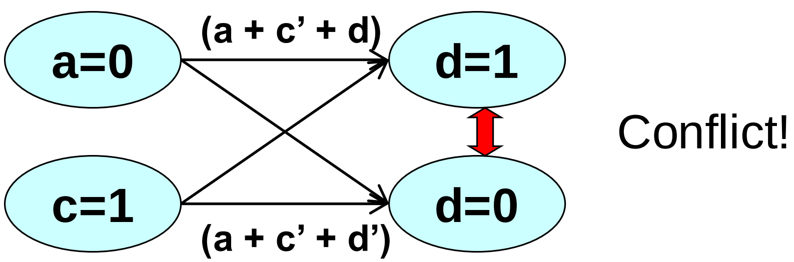

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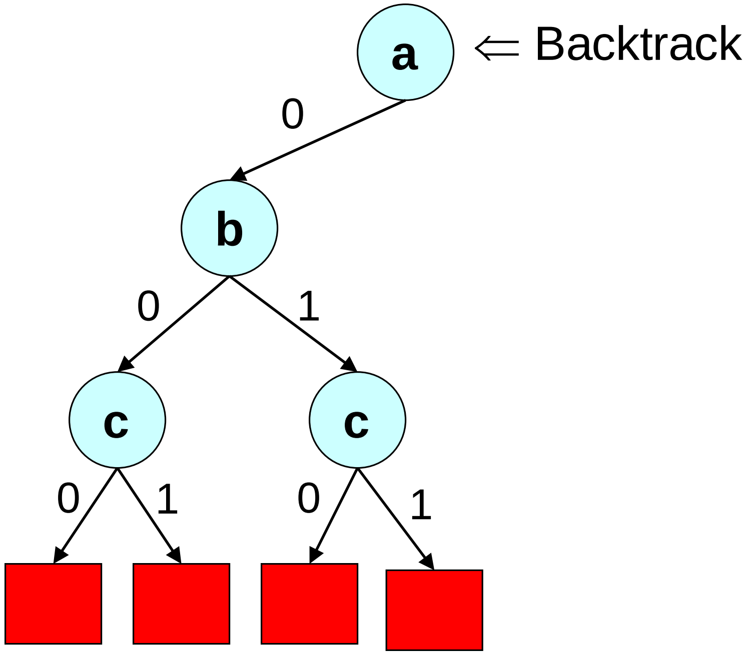

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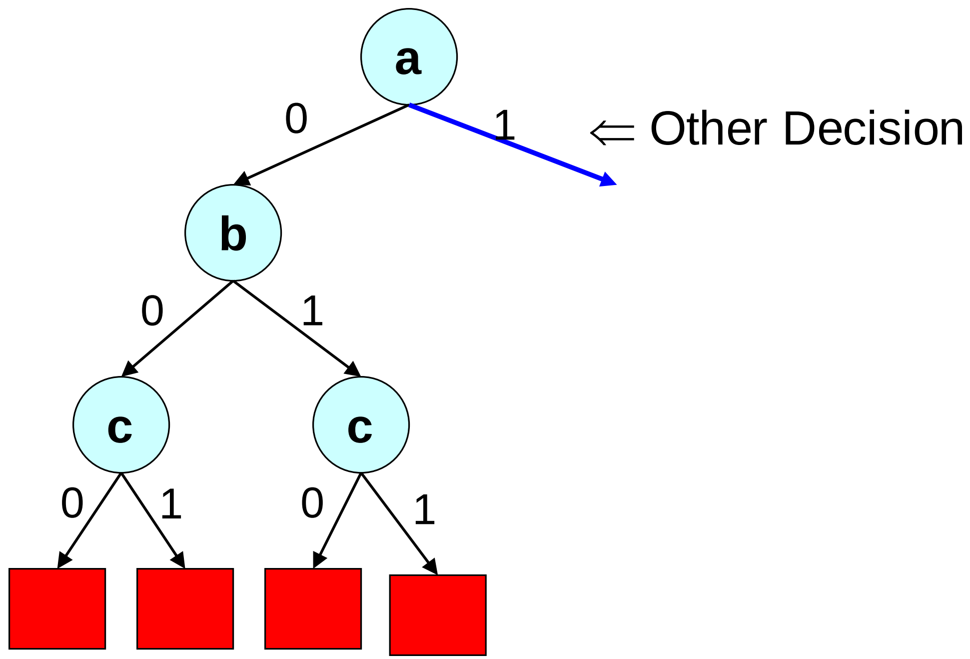

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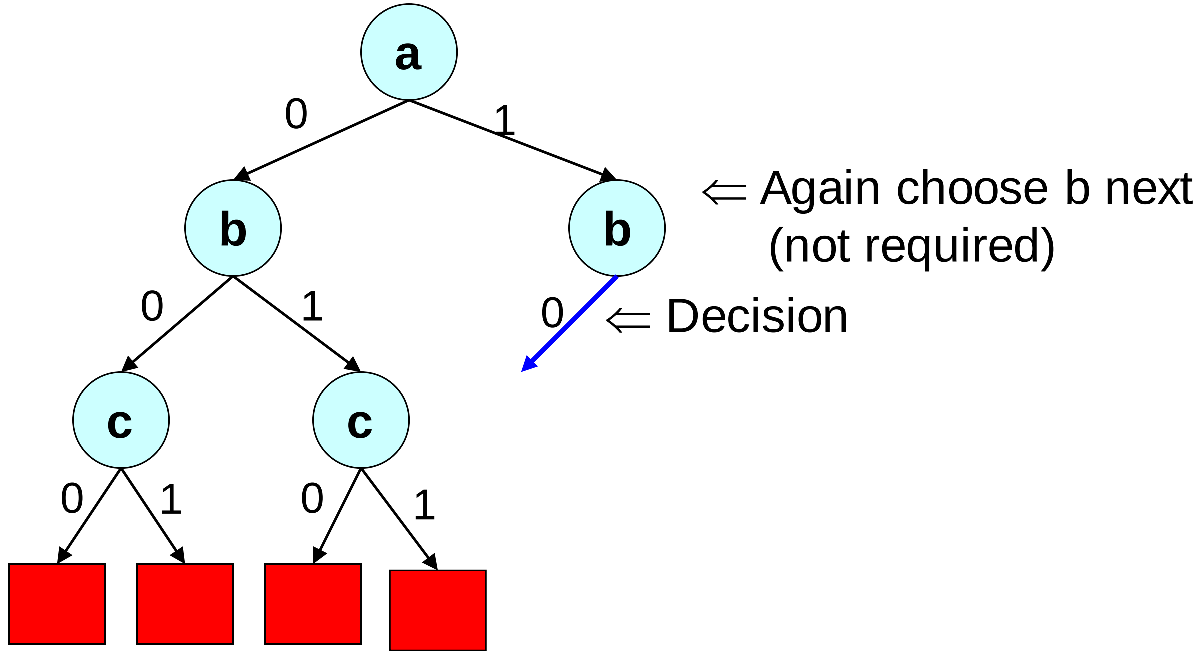

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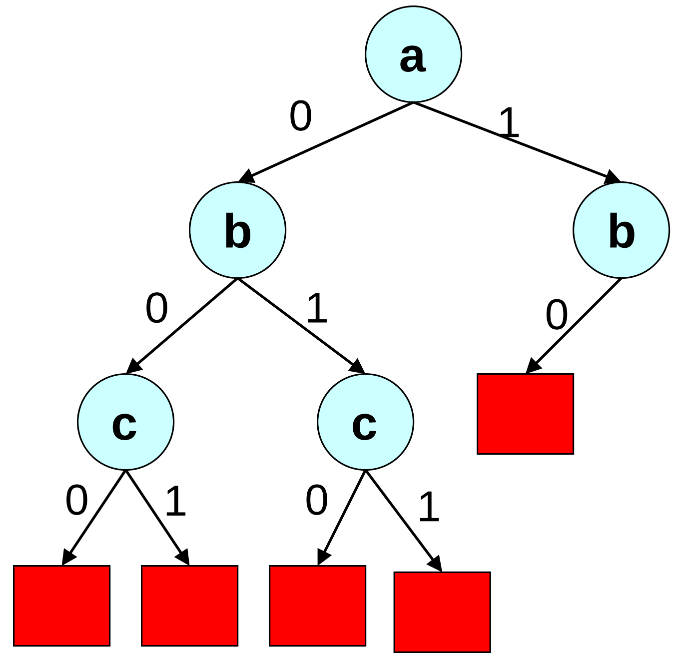

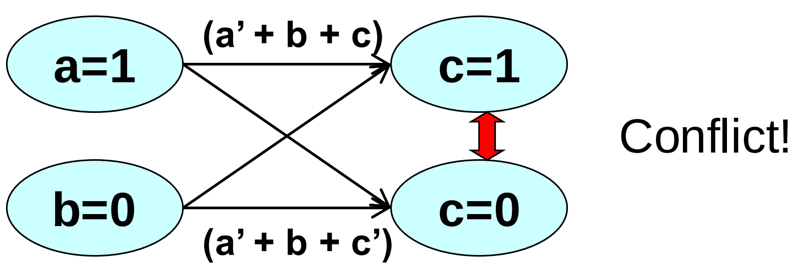

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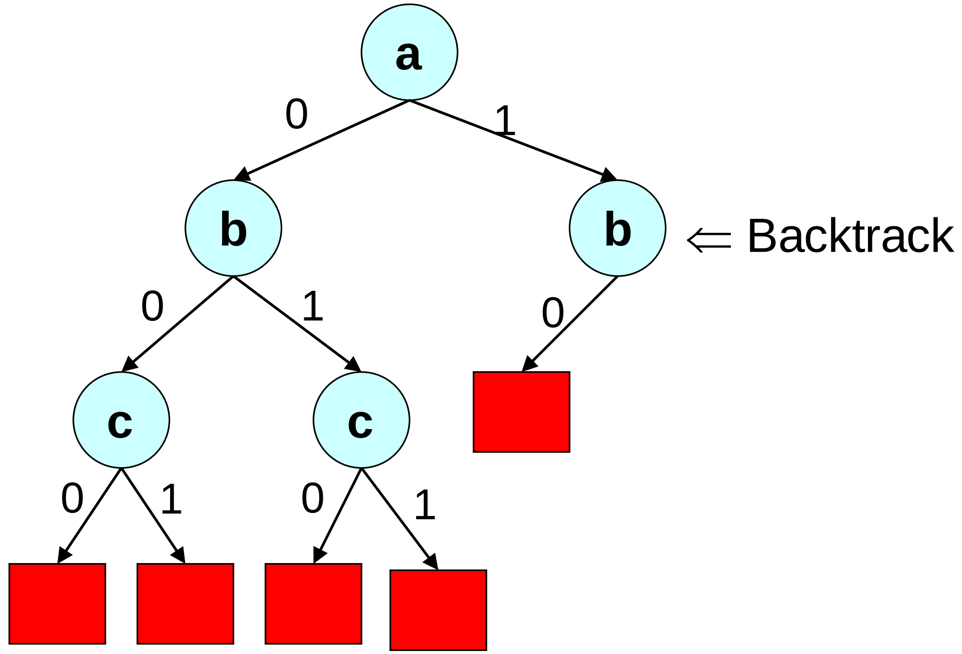

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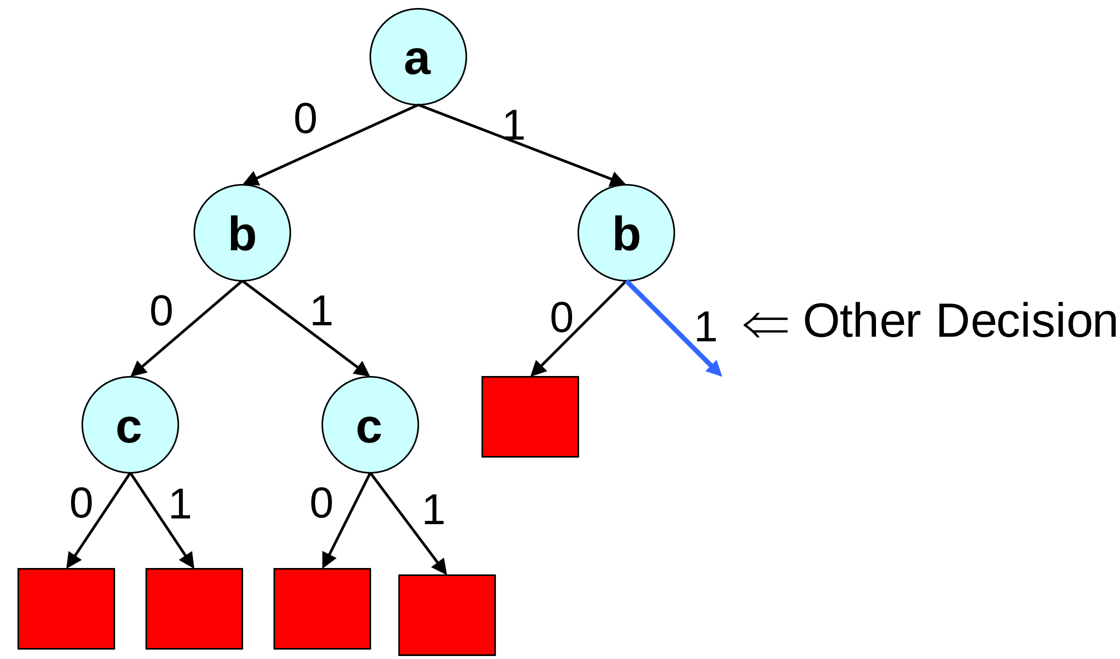
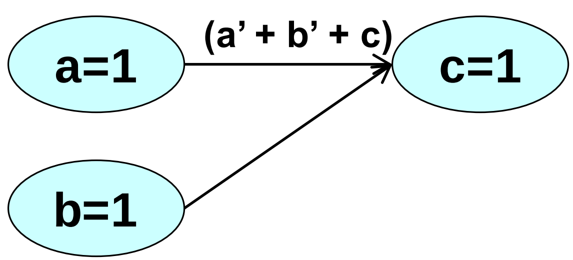


unit clause that propagates without contradiction (finally!) Often you get these much sooner



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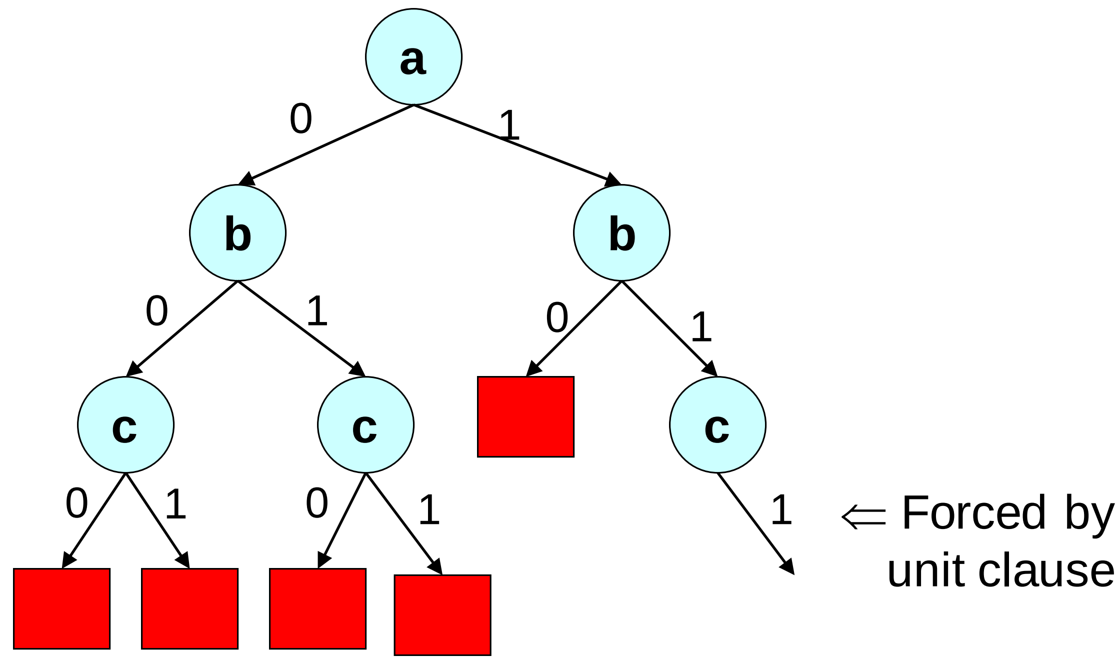

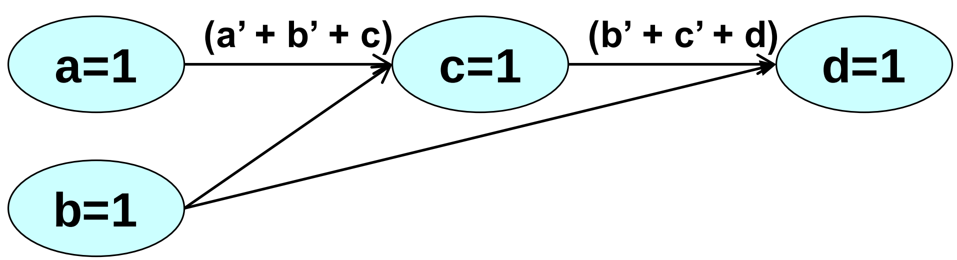

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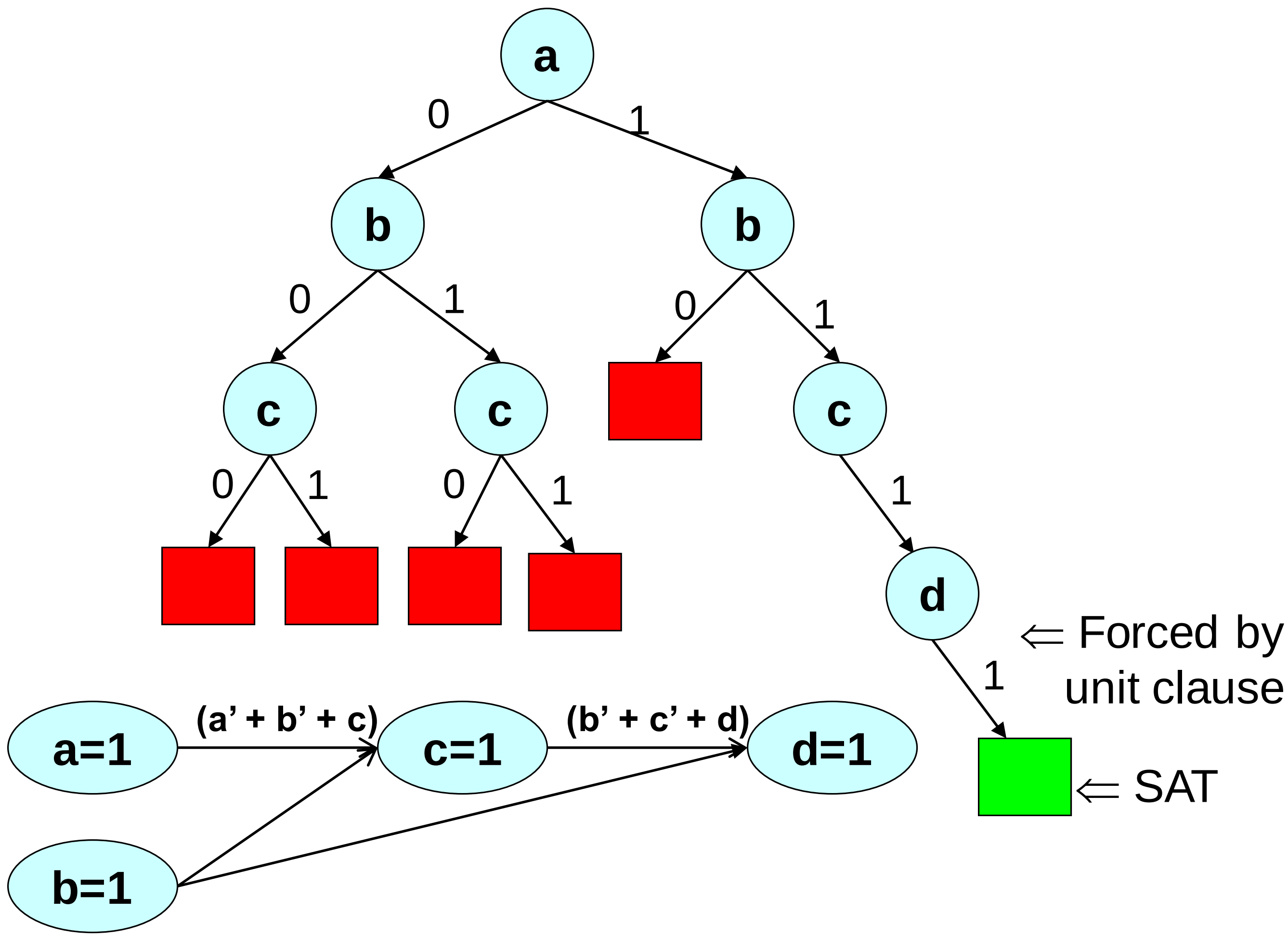


slide thanks to Sharad Malik (modified)

Tricks used by zChaff and similar DLL solvers

(Overview only; details on later slides)

- **Make unit propagation / backtracking speedy** (80% of the cycles!)
- **Variable ordering heuristics:** Which variable/value to assign next?
- **Conflict analysis:** When a contradiction is found, analyze what subset of the assigned variables was responsible. Why?
  - **Better heuristics:** Like to branch on vars that caused recent conflicts
  - **Backjumping:** When backtracking, avoid trying options that would just lead to the same contradictions again.
  - **Clause learning:** Add new clauses to block bad sub-assignments.
  - **Random restarts** (maybe): Occasionally restart from scratch, but keep using the learned clauses. (Example: crosswords ...)
    - Even without clause learning, random restarts can help by abandoning an unlucky, slow variable ordering. Just break ties differently next time.
- **Preprocess** the input formula (maybe)
- **Tuned implementation:** Carefully tune data structures
  - improve memory locality and avoid cache misses

Motivating Metrics: Decisions, Instructions, Cache Performance and Run Time

	1dlx_c_mc_ex_bp_f
Num Variables	776
Num Clauses	3725
Num Literals	10045

	Z-Chaff	SATO	GRASP
# Decisions	3166	3771	1795
# Instructions	86.6M	630.4M	1415.9M
# L1/L2 accesses	24M / 1.7M	188M / 79M	416M / 153M
% L1/L2 misses	4.8% / 4.6%	36.8% / 9.7%	32.9% / 50.3%
# Seconds	0.22	4.41	11.78

slide thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0						

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0						

= forced by propagation

= first guess

= second guess

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0			

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0						

Guess a new assignment J=0

(white)	= forced by propagation
(yellow)	= first guess
(red)	= second guess

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0			

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0	K=1	L=1				

Unit propagation implies assignments K=1, L=1

- = forced by propagation
- = first guess
- = second guess
- = currently being propagated
- = assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0	1		

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0	K=1	L=1				

Now make those assignments, one at a time

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0	1		

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0			

Chain reaction: K=1 propagates to imply B=0

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0	1		

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0		

Also implies A=1, but we already knew that

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0	1	1	

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0			

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0	1	1	

Stack of assignments used for backtracking

Oops!

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0	F=1	

L=1 propagates to imply F=1, but we already had F=0

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0	1	1	

Stack of assignments used for backtracking

Oops!

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0	F=1	

Backtrack to last yellow, undoing all assignments

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			0			

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=0						

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			1			

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=1						

J=0 didn't work out, so try J=1

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1		1			0	0			1			

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=1	B=0				

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1	0	1			0	0			1			

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=1	B=0					

Nothing left to propagate. Now what?

- = forced by propagation
- = first guess
- = second guess
- = currently being propagated
- = assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1	0	1			0	0			1		1	

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=1	B=0	L=1	…			

Again, guess an unassigned variable and proceed …

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1	0	1			0	0			1		0	

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=1	B=0	L=0	…		

If L=1 doesn't work out, we know L=0 in this context

= forced by propagation

= first guess

= second guess

= currently being propagated

= assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1	0	1			0	0			1		0	

Stack of assignments used for backtracking

C=1	F=0	A=1	G=0	J=1	B=0	L=0	…			

If L=0 doesn't work out either, backtrack to … ?

- = forced by propagation
- = first guess
- = second guess
- = currently being propagated
- = assignment still pending

DLL: Obvious data structures

Current variable assignments

A	B	C	D	E	F	G	H	I	J	K	L	M
1					1							

Stack of assignments used for backtracking

C=1	F=1									

Question: When should we return SAT or UNSAT?

= forced by propagation

= first guess

= second guess

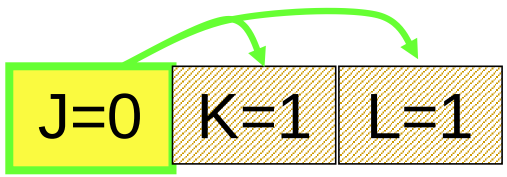
= currently being propagated

= assignment still pending

How to speed up unit propagation?

(with a million large clauses to keep track of)

Every step in DLL is fast, except propagation:



(Yes, even picking next unassigned variable is fast, if we don't try to be smart about it.)

- Objective: When a variable is assigned to 0 or 1, detect which clauses become unit clauses.
  - Obvious strategy: "crossing out" as in previous slides. Too slow, especially since you have to un-cross-out when backtracking.
  - Better: Don't modify or delete clauses. Just search for k-clauses with **k-1 currently false** literals & **1 currently unassigned** literal.
  - But linear search of all the clauses is too slow.

Sounds like grid method!

How to speed up unit propagation?

(with a million large clauses to keep track of)

- Find length-k clauses with **k-1 false** literals & **1 unassigned** literal.

- **Index** the clauses for fast lookup:
  - Every literal (A or ~A) maintains a list of clauses it's in
  - If literal becomes false, only check if <u>those</u> clauses became unit
  - Could use **counters** so that checking each clause is fast:
    - Every clause remembers how many **non-false** literals it has
    - If this counter ever gets to 1, we might have a new unit clause
      - Scan clause to find the remaining non-false literal
      - It's either true, or unassigned, in which case we assign it true!
    - When variable A is assigned, either A or ~A becomes false
      - Increment counters of all clauses containing the newly false literal
      - When undoing the assignment on backtracking, decrement counters

Too many clauses to visit!
(worse, a lot of memory access)

How to speed up unit propagation?

(with a million large clauses to keep track of)

- Find length-k clauses with **k-1 false** literals & **1 unassigned** literal.
    - When variable A is assigned, either A or ~A becomes false
        - Decrement counters of all clauses containing the newly false literal
        - Clause only becomes unit when its counter reaches 1

Too many clauses to visit!

- **Hope:** Don't keep visiting clause just to adjust its counter.
    - So, can't afford to keep a counter.
    - Visit clause only when it's <u>really</u> in danger of becoming unit.
- **Insight:** A clause with at least 2 non-false literals is safe.
    - So pick any 2 non-false literals (true/unassigned), and watch them.
    - As long as both **stay** non-false, don't have to visit the clause at all!
- **Plan:** Every literal maintains a list of clauses in which it's <u>watched</u>.
    - If it becomes false, we go check those clauses (only).

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E

```
( B  C  A  D  E)
( A  B ~C)
( A ~B)
(~A  D)

(~A)
```

Always watch first 2 literals in each clause

How about clauses with only one literal?

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E

A=0				

```
( B   C   A   D   E)
( A   B  ~C)
( A  ~B)
(~A   D)
```

Always watch first 2 literals in each clause

Invariant: Keep false vars out of these positions as long as possible.
Why? Watched positions **must** be the **last** ones to become false - so that we'll notice when that happens!

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0				

A=0				

```
( B  C  A  D  E)
( A  B ~C)
( A ~B)
(~A  D)
```

Not watched! Can get assigned/unassigned many times for free.

Only cares if it's ~A that becomes false.

A became false. It is watched only here.
Look to see if either of these 2 clauses became unit.

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0				

A=0				

(B C A D E)

(A B ~C)

(A ~B)

(~A D)

A became false. It is watched only here.
Look to see if either of these 2 clauses became unit.

(A B ~C) is now (0 ? ?), so not unit yet.
To keep 0 vars out of watched positions, swap A with ~C.
(So if ~C is the last to become false, giving a unit clause, we'll notice.)

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0				

A=0	B=0			

```
( B  C  A  D  E)
(~C  B  A)
( A ~B)
(~A  D)
```

A became false. It is watched only here.
Look to see if either of these 2 clauses became unit.

(A ~B) is now (0 ?), so unit.
We find that ~B is the unique ? variable, so we must make it true.

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0			

A=0	$\boxed{\text{B=0}}$			

```
( B  C   A  D  E)
(~C  B   A)
( A ~B)
(~A  D)
```

B became false. It is watched only here.
Look to see if either of these 2 clauses became unit.

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0			

A=0	B=0			

```
( B  C  A  D  E)
(~C  B  A)
( A ~B)
(~A  D)
```

B became false. It is watched only here.
Look to see if either of these 2 clauses became unit.

(B C A D E) is now (0 ? 0 ? ?), so not unit yet.
To keep 0 vars out of watched positions, swap B with D.
(So if D is the last to become false, giving a unit clause, we'll notice.)

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0			

A=0	B=0	C=0		

```
( D  C   A   B   E)
(~C  B   A)
( A ~B)
(~A  D)
```

B became false. It is watched only here.
Look to see if either of these 2 clauses became unit.

(~C B A) is now (? 0 0), so unit.
We find that ~C is the unique ? variable, so we must make it true.

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0	0		

A=0	B=0	C=0		

```
( D  C  A  B  E)
(~C  B  A)
( A ~B)
(~A  D)
```

C became false. It is watched only here.
Look to see if this clause became unit.

(D C A B E) is now (? 0 0 0 ?), so not unit yet.
To keep 0 vars out of watched positions, swap C with E.
(So if E is the last to become false, giving a unit clause, we'll notice.)

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0	0	1	

A=0	B=0	C=0	D=1	

```
( D   E   A   B   C)
(~C   B   A)
( A  ~B)
(~A   D)
```

We decided to set D true.
So ~D became false … but it's not watched anywhere, so nothing to do.

(First clause became satisfied as (1 ? 0 0 0), but we haven't noticed yet. In fact, we haven't noticed that **all** clauses are now satisfied!)

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0	0	1	0

A=0	B=0	C=0	D=1	E=0

```
( D  E  A  B  C)
(~C  B  A)
( A ~B)
(~A  D)
```

We decided to set E false. It is watched only here.
Look to see if this clause became unit.

(D E A B C) is now (1 0 0 0 0). This is not a unit clause. (It is satisfied because of the 1.)

All variables have now been assigned without creating any conflicts, so we are SAT.

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0	0		

A=0	B=0	C=0		

```
( D  C  A  B  E)
(~C  B  A)
( A ~B)
(~A  D)
```

- Why the technique is fast:
  - Assigning 0 to a watched literal takes work
    - Have to check for unit clause, just in case
    - Must try to move the 0 out of watched position
  - But everything else costs nothing …
    - Assigning 1 to a watched literal
    - Unassigning a watched literal (whether it's 0 or 1)
    - Doing <u>anything</u> to an <u>unwatched</u> literal.

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0	0		

A=0	B=0	C=0		

```
( D  C  A  B  E)
(~C  B  A)
( A ~B)
(~A  D)
```

- Why it's even faster than you realized:
  - Assigning 0 to a watched literal takes work
    - But then we'll move it out of watched position if we can!
    - So <u>next</u> time we try to assign 0 to the same literal, it costs nothing!
      - This is very common: backtrack to change B, then retry C=0
    - (**Deep analysis:** Suppose a clause is far from becoming a unit clause (it has a few true or unassigned vars). Then we shouldn't waste much time repeatedly checking whether it's become unit. And this algorithm doesn't: Currently "active" variables get swapped out of the watch positions, in favor of "stable" vars that are true/unassigned. "Active" vars tend to exist because we spend most of our time shimmying locally in the search tree in an inner loop, trying many assignments for the last few decision variables that lead to a conflict.)

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Chaff/zChaff's unit propagation algorithm

A	B	C	D	E
0	0	0		

A=0	B=0	C=0		

```
( D  C  A  B  E)
(~C  B  A)
( A ~B)
(~A  D)
```

- There is a bit of overhead.
- Each literal maintains an array of clauses watching it.
- When C becomes 0, we iterate through its array of clauses:
  - We scan clause 1: discover it's not unit, but we must swap C, E.
  - So take clause 1 off C's watch list and add it to E's watch list.
  - Not hard to make this fast (see how?)

example thanks to Moskewicz, Madigan, Zhang, Zhao, & Malik

Big trick #2: Conflict analysis

- When a contradiction is found, analyze what subset of the assigned variables was responsible. Why?
  - **Backjumping:** When backtracking, avoid trying options that would just lead to the same contradictions again.
  - **Clause learning:** Add new clauses to block bad sub-assignments.
  - **Random restarts:** Occasionally restart from scratch, but keep using the learned clauses (or try a new variable ordering).
  - **Better heuristics:** Like to branch on vars that caused recent conflicts

Each var records its decision level

A	B	C	D	E	F	G	H	I	J	K
2		1			2	2			3	3
1		1			0	0			0	1

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0
1	2	2	2	3	3		

Decision levels

Big trick #2: Conflict analysis

- Each var also remembers **why** it became 0 or 1
  - A yellow (or red) var remembers it was a decision variable
  - A white var was set by unit propagation
    - Remembers which clause was responsible
    - That is, points to the clause that became a unit clause & set it

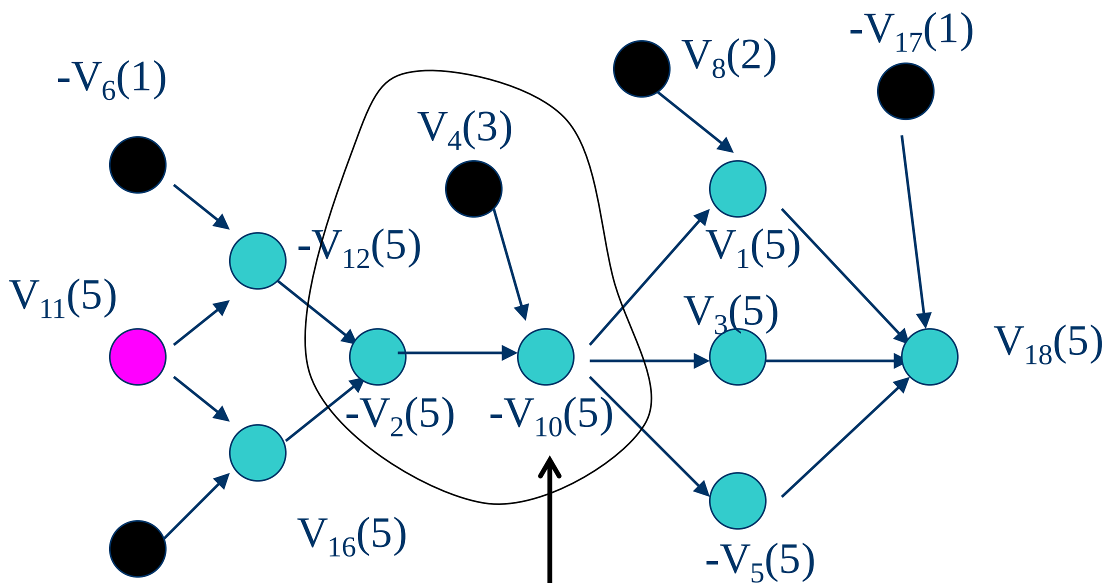
Each var records its decision level

A	B	C	D	E	F	G	H	I	J	K
2		1			2	2			3	3
1		1			0	0			0	1

C=1	F=0	A=1	G=0	J=0	K=1	L=1	B=0
1	2	2	2	3	3		

Decision levels

Can regard those data structures as an "implication graph"



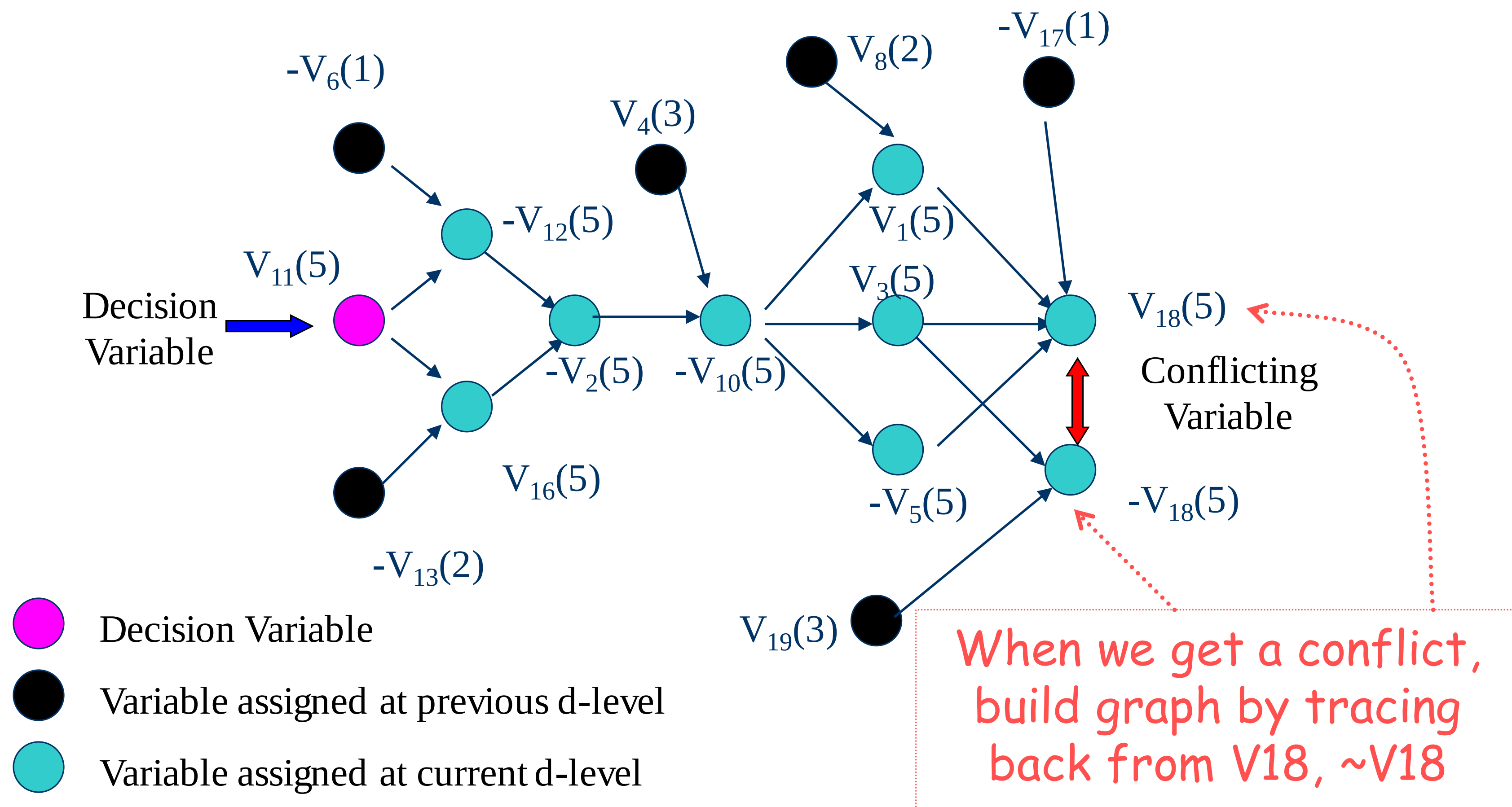
E.g., remember, variable V_{10} stores a pointer saying why V_{10}=0:

because (V_4'+V_2+V_{10}') became a unit clause at level 5

In other words, V_{10}=0 because V_4=1 **and** V_2=0: we draw this!

slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

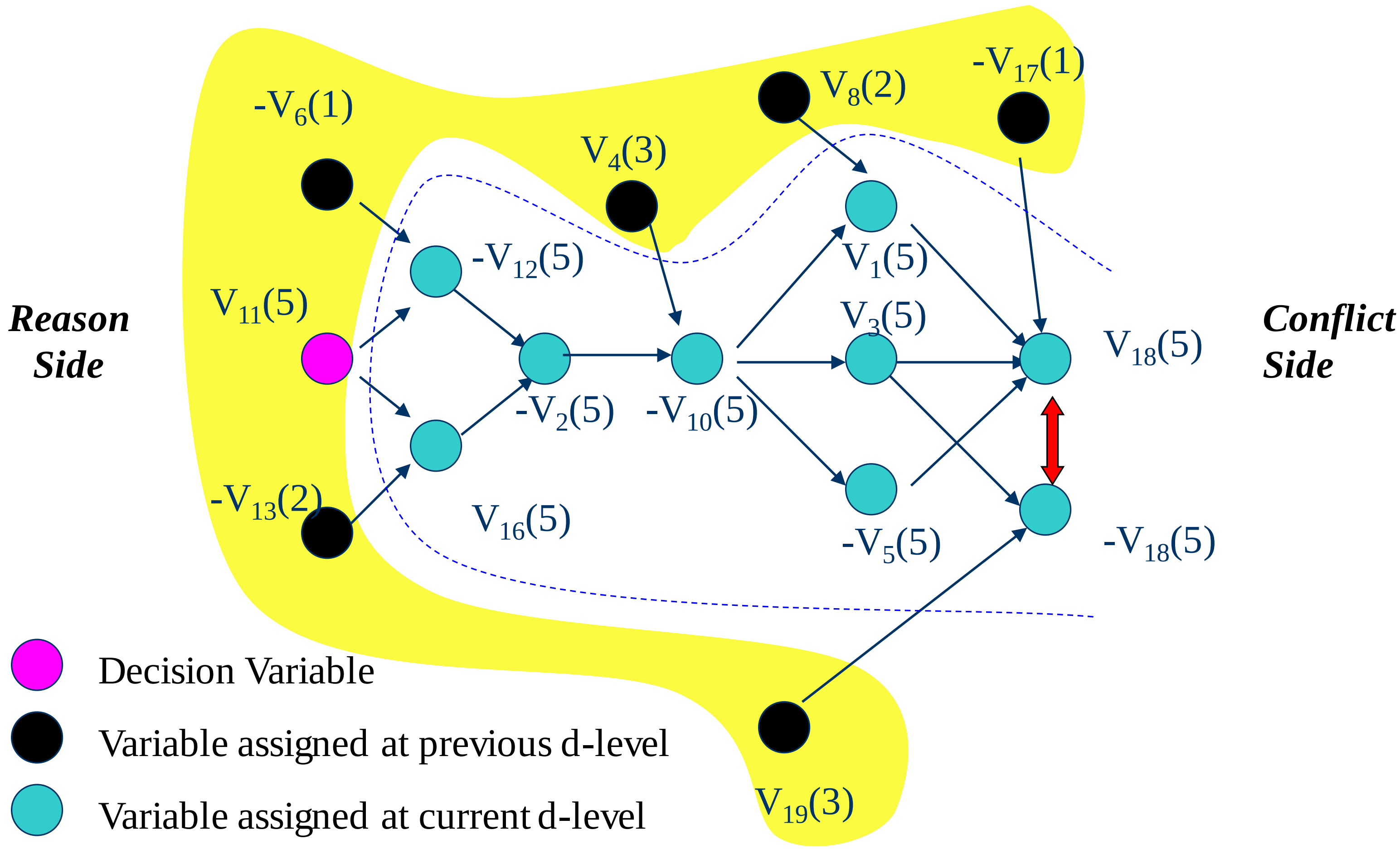
Can regard those data structures as an "implication graph"



slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

Which decisions triggered the conflict?

(in this case, only decisions 1,2,3,5 – not 4)



slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

Which decisions triggered the conflict?

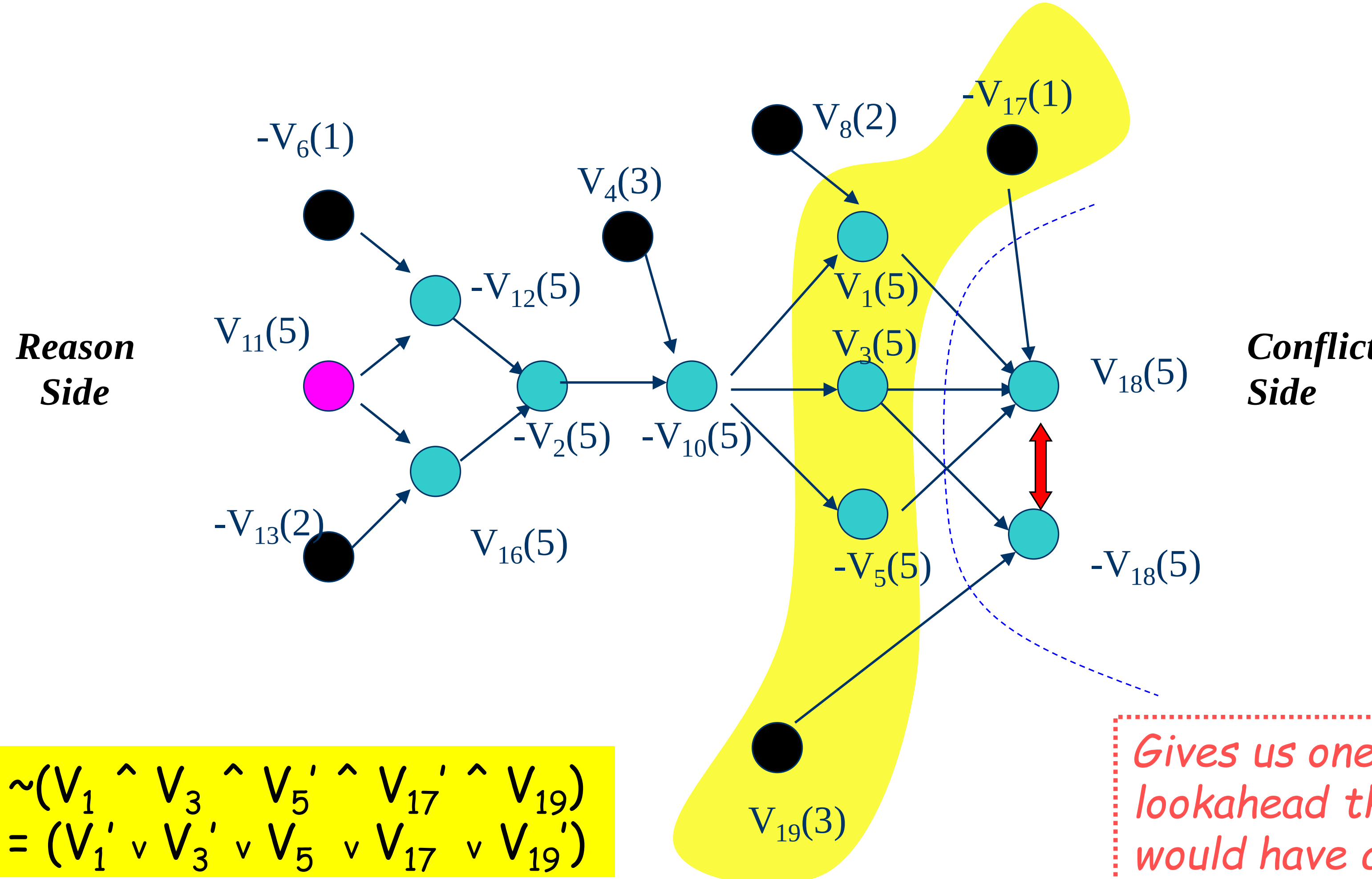
(in this case, only decisions 1,2,3,5 – not 4)

- Our choices at decision levels 1,2,3,5 were jointly responsible
- So decision level 4 was **not** responsible
  - Neither decision <u>variable</u> 4, nor any vars set by propagation from it
- Suppose we now backtrack from decision 5
  - We just found =1 led to conflict; suppose we found =0 did too
  - And suppose level 4 wasn't responsible in **either** case
  - Then we can "backjump" over level 4 back to level 3
    - Trying other value of decision variable 4 can't avoid conflict!
    - Also called "non-chronological backtracking"
    - Should now copy conflict information up to level 3: all the choices at levels 1,2,3 were responsible for this conflict below level 3

slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

A possible clause to learn

(avoid 5-variable combination that will always trigger this V_{18} conflict)



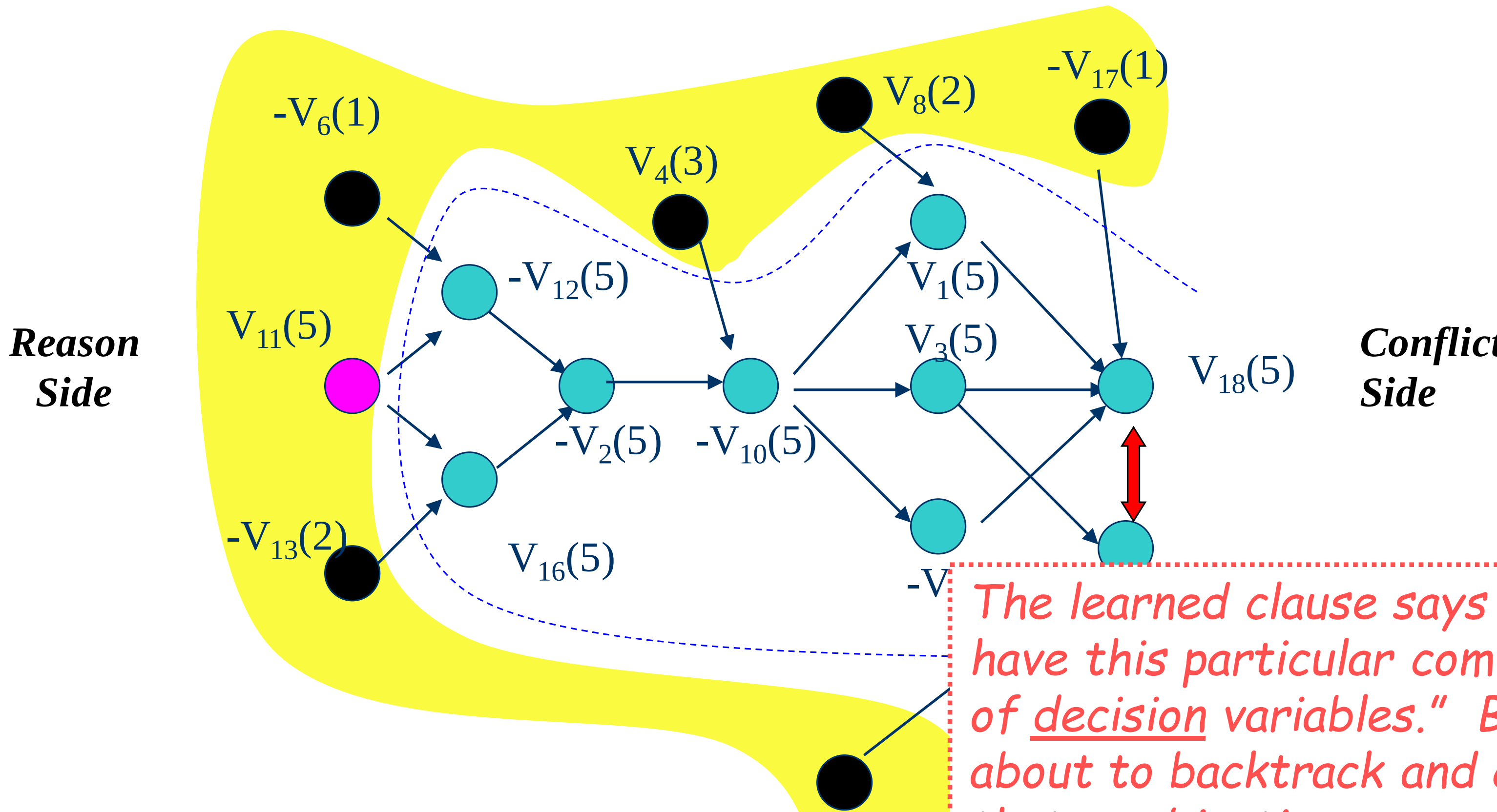
$\sim(V_1 \wedge V_3 \wedge V_5' \wedge V_{17}' \wedge V_{19})$
$= (V_1' \vee V_3' \vee V_5 \vee V_{17} \vee V_{19}')$

Gives us one step of lookahead that would have avoided this conflict

slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

A different clause we could learn

(would have caught the problem sooner: but why not as great an idea?)



The learned clause says "don't have this particular combination of decision variables." But we're about to backtrack and change that combination anyway – so the learned clause won't be useful for a while.

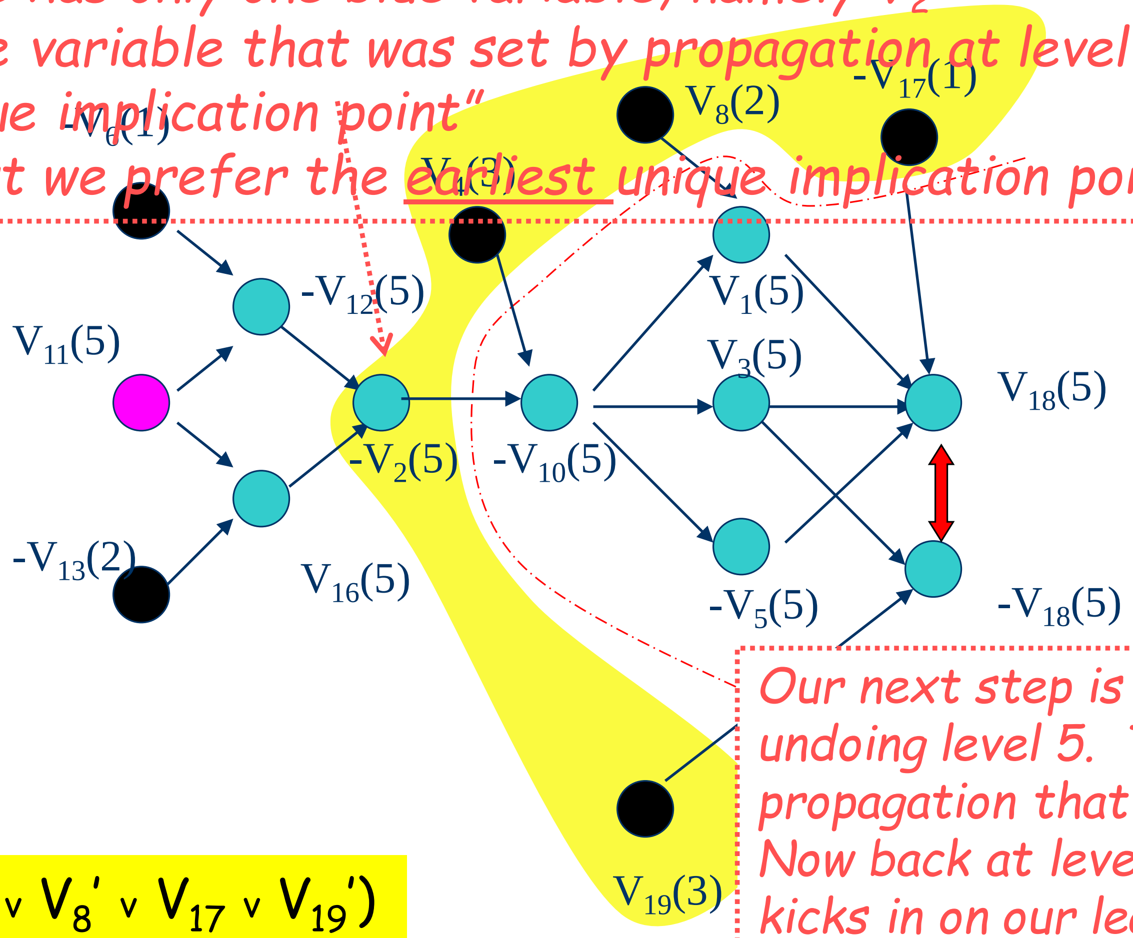
$(V_2 \vee V_4' \vee V_6 \vee V_8' \vee V_2 \vee V_{11}' \vee V_{13} \vee V_{17} \vee V_{19}')$

slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

A different clause we could learn

(would have caught the problem sooner)

Learned clause has only one blue variable, namely V_2
(i.e., only one variable that was set by propagation at level 5)
V_2 is a "unique implication point"
(so is V_{10}, but we prefer the <u>earliest</u> unique implication point)



$(V_2 \vee V_4' \vee V_8' \vee V_{17} \vee V_{19}')$

Our next step is to backtrack, undoing level 5. This undoes the propagation that set V_2=0. Now back at level 4, propagation kicks in on our learned clause, forcing V_2=1 (and maybe more!).

slide thanks to Zhang, Madigan, Moskewicz, & Malik (modified)

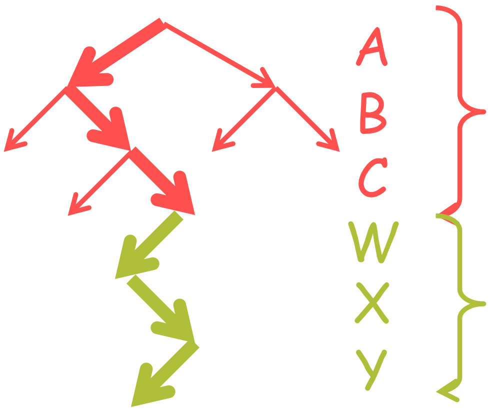
Variable ordering heuristics

- Function DLL(φ):
  - do unit propagation
  - **if** we got a conflict, return UNSAT
  - **else if** all variables are assigned, return SAT
  - **else**
    - **pick an unassigned variable X**
    - **if** DLL(φ ^ X) = SAT **or** DLL(φ ^ ~X) = SAT
    - **then** return SAT **else** return UNSAT

- How do we choose the next variable X? (variable ordering)
  - And how do we choose whether to try X or ~X first? (value ordering)
- Would like choice to lead quickly to a satisfying assignment …
- … or else lead quickly to a conflict so we can backtrack.
- Lots of heuristics have been tried!

Heuristic: Most Constrained First

- Pack suits/dresses before toothbrush
  - First try possibilities for highly constrained variables
  - Hope the rest fall easily into place

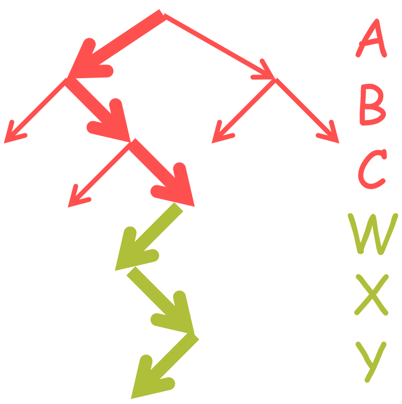


First find an assignment for the most constrained variables ABC (often hard, much backtracking)

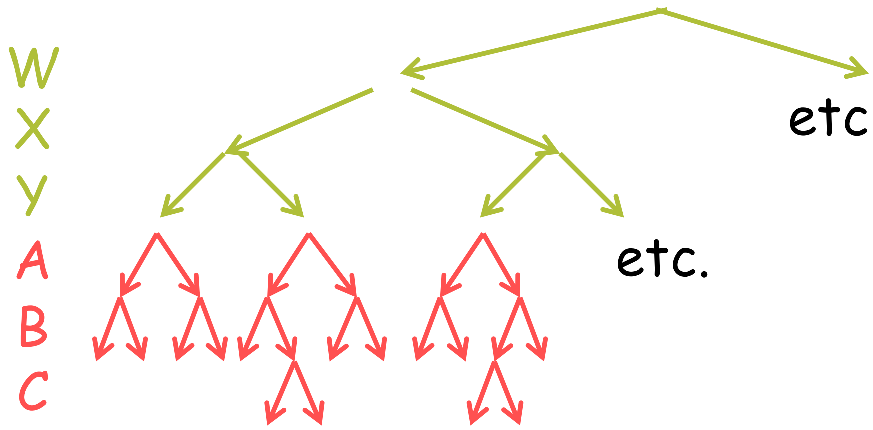
If that succeeds, may be easy to extend it into a full assignment: other vars WXY fall easily into place (e.g., by unit propagation from vars already chosen, or because either W=0 or W=1 could lead to success)

Heuristic: Most Constrained First

- Pack suits/dresses before toothbrush
  - First try possibilities for highly constrained variables
  - Hope the rest fall easily into place



Good: solve for most constrained first



Bad: start with least constrained. All assignments to WXY start out looking good, but then we usually can't extend them.

Heuristic: Most Constrained First

- **So, how do we guess which vars are "most constrained"?**
- Which variable appears in the most clauses?
- Wait: shorter clauses are stronger constraints
  - (A B) rules out 1/4 of all solutions
  - (A B C D E) only rules out 1/32 of all solutions; doesn't mean A is especially likely to be true
- Which variable appears in the most "short clauses"?
  - E.g., consider clauses only of minimum length

??

- As we assign variables, "crossing out" will shorten or eliminate clauses. Should we consider that when picking next variable?
  - "Dynamic" variable ordering – depends on assignment so far
  - Versus "fixed" ordering based on original clauses of problem
  - Dynamic can be helpful, but you pay a big price in bookkeeping

slide thanks to Ian Gent (modified)

Heuristic: Most satisfying first

(Jeroslow-Wang heuristic for variable and value ordering)

- Greedily satisfy (eliminate) as many clauses as we can
  - Which literal appears in the most clauses?
  - If X appears in many clauses, try X=1 to eliminate them
  - If ~X appears in many clauses, try X=0 to eliminate them
- Try especially to satisfy hard (short) clauses
  - When counting clauses that contain X,
  - length-2 clause counts twice as much as a length-3 clause
  - length-3 clause counts twice as much as a length-4 clause
  - In general, let a length-i clause have weight 2^{-i}
    - Because it rules out 2^{-i} of the 2^{-n} possible assignments

slide thanks to Ian Gent (modified)

Heuristic: Most simplifying first

(again, does variable and value ordering)

- We want to simplify problem as much as possible
  - I.e. get biggest possible cascade of unit propagation
  - Motivation: search is exponential in the size of the problem so making the problem small quickly minimizes search
- One approach is to try it and see
  - make an assignment, see how much unit propagation occurs
  - after testing all assignments, choose the one which caused the biggest cascade
  - exhaustive version is expensive (2n probes necessary)
  - Successful variants probe a small number of promising variables (e.g. from the "most-constrained" heuristic)

slide thanks to Ian Gent (modified)

Heuristic: What does zChaff use?

- Use variables that appeared in many “recent” learned clauses or conflict clauses
  - Keep a count of appearances for each variable
  - Periodically divide all counts by a constant, to favor recent appearances

- Really a subtler kind of “most constrained” heuristic
  - Look for the “active variables” that are currently hard to get right

- Definitely a dynamic ordering … but fast to maintain
  - Only update it once per conflict, not once per assignment
  - Only examine one clause to do the update, not lots of them

Random restarts

- Sometimes it makes sense to keep restarting the search, with a different variable ordering each time
  - Avoids the very long run-times of unlucky variable ordering
  - On many problems, yields faster algorithms

- Why would we get a different variable ordering?
  - We break ties randomly in the ordering heuristic
    - Or could add some other kind of randomization to the heuristic
  - Clauses learned can be carried over across restarts
    - So after restarting, we're actually solving a different formula

slide thanks to Tuomas Sandholm