"hysteresis," caused by the subcritical bifurcation. The bifurcation diagrams for the backstepping controller (57) are shown in Fig. 4 for $c_0 = 6$ and $\bar{\beta}^2 k = 0.5$. This controller "softens" the bifurcation from subcritical to supercritical and eliminates the hysteresis. In addition, it stabilizes all stall equilibria and prevents surge for all values of Γ.

While all three designs in Table I soften the bifurcation, the global design achieved with backstepping is due to a methodological difference. The bifurcation designs in [14] and [7] are based on local stability properties established by the center manifold theorem because the maximum of the compressor characteristic is a bifurcation point that is not linearly controllable. Hence stabilization is inherently nonlinear and results in asymptotic but not exponential stability. Our Lyapunov-based design incorporates the good features of a bifurcation-based design. For $\Phi_0 = 1$, the term $V_r(r)$ in the Lyapunov function (49), becomes $V_r(R) = R$, so that the Lyapunov function

$$V_2 = \frac{c_0}{2}\left[\left(c_1 + \frac{3}{8} + \frac{1}{c_0\beta^2}\right)\left(\phi^2 + \frac{5}{\sigma}R\right) + \frac{9}{8}\phi^2 + \phi^2 + \frac{1}{4}\phi^4 + 3R\phi^2\right] + \frac{1}{2}(\psi - c_0\phi)^2 \qquad (59)$$

is (locally) quadratic in ϕ and ψ but only linear in R. Since the derivative of V_2 is quadratic in all three variables, $\dot{V}_2 \le -a_1\phi^2 - a_2R^2 - a_3(\psi - c_0\phi)^2$ [see (56)], this clearly indicates that the achieved type of stability is asymptotic but not exponential. However, to satisfy the requirements not only for local but also for global stability, our analysis is considerably more complicated.

V. Conclusion

Experimental validation of the controller presented here is planned but is beyond the scope of this paper. Measurement of Φ represents a challenge but it is not expected to be insurmountable considering that a controller that employs the *derivative* of Φ has been successfully implemented [7].

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A Deterministic Analysis of Stochastic Approximation with Randomized Directions

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***Abstract*— We study the convergence of two stochastic approximation algorithms with randomized directions: the simultaneous perturbation stochastic approximation algorithm and the random direction Kiefer–Wolfowitz algorithm. We establish deterministic necessary and sufficient conditions on the random directions and noise sequences for both algorithms, and these conditions demonstrate the effect of the "random" directions on the "sample-path" behavior of the studied algorithms. We discuss ideas for further research in analysis and design of these algorithms.**

***Index Terms*—Deterministic analysis, random directions, simultaneous perturbation, stochastic approximation.**

I. Introduction

One of the most important applications of stochastic approximation algorithms is in solving local optimization problems. If an estimator of the gradient of the criterion function is available, the Robbins–Monro algorithm [9] can be directly applied. In [7], Kiefer

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and Wolfowitz present a modification of the standard Robbins–Monro algorithm to recursively estimate the extrema of a function in the scalar case. The algorithm is based on a finite-difference estimate of the gradient and does not require knowledge of the structure of the criterion function. In [1], Blum presents a multivariate version of Kiefer and Wolfowitz's algorithm for higher dimensional optimization. We will refer to this algorithm as the Kiefer–Wolfowitz (KW) algorithm. For an objective function with dimension p, the finite-difference estimation generally requires $2p$ observations at each iteration. This requirement usually results in unrealistic computational complexity when the dimension of the problem is high. To circumvent this problem, several authors have studied variants of the KW algorithm based on finite-difference estimates of the directional derivatives along a sequence of randomized directions; see, for example, [6], [8], and [12]. We will refer to this type of algorithm as the *random direction Kiefer–Wolfowitz* (RDKW) algorithm. The number of observations needed by the RDKW algorithm is two per iteration, regardless of the dimension of the problem. However, the question arises as to whether the increase in the number of iterations (due to the randomization of the directions) may offset the reduction in the amount of data per iteration, resulting in worse overall performance. Different distributions for the randomized direction sequence have been considered: uniform distribution in [6], spherically uniform distribution in [8], and Normal and Cauchy distributions in [12]. None of these results theoretically establish the superiority of the RDKW algorithms with respective direction distribution over the standard KW algorithm.

In [11], Spall presents a KW-type algorithm based on a "simultaneous perturbation" gradient approximation that requires only two observations at each iteration. The algorithm also moves along a sequence of randomized directions as the RDKW algorithm. By analyzing the asymptotic distribution, Spall [11] shows that the proposed algorithm can be significantly more efficient than the standard KW procedure. Following the terminology of [11], we refer to Spall's algorithm as the *simultaneous perturbation stochastic approximation* (SPSA) algorithm. Chin presents in [4] both theoretical and numerical comparison of the performance of KW, RDKW (the name RDSA is used therein), and SPSA algorithms. A more general class of distributions is considered for the RDKW algorithm there and the SPSA algorithm is shown to exhibit the smallest asymptotic mean squared error. Chin makes an assumption that components of each direction have unity variance, which is not necessary for convergence of the RDKW algorithm as illustrated by Proposition 2. In fact, as explained in Section IV, we can show that the SPSA and RDKW algorithms achieved the same level of performance asymptotically under optimized conditions. In [2], Chen *et al.* study a modification of the SPSA algorithm and prove its convergence under weaker conditions.

In this paper, we focus on the sample-path analysis of the SPSA and the RDKW algorithms. We develop a deterministic analysis of the algorithms and present deterministic necessary and sufficient conditions on both the randomized directions and noise sequence for convergence of these algorithms. Different from the results in [2], [4], and [11], we treat the "randomized" direction sequence as an arbitrary deterministic sequence and derive the conditions on each individual sequence for convergence of these algorithms. The resulting condition displays the sample-path effect of each random direction sequence on the convergence of both algorithms.

Throughout the paper, we consider the problem of recursively estimating the minimum of an objective function L: $R^p \to R$ based on noisy measurements of L. We assume that L satisfies the following conditions.

A1) The gradient of L, denoted by $f = \nabla L$, exists and is uniformly continuous.

A2) There exist $x^* \in R^p$ such that

- $f(x^*) = 0$;
- for all $\delta > 0$, there exists $h_\delta > 0$ such that $\|x - x^*\| \geq \delta$ implies $f(x)^T(x - x^*) \geq h_\delta \|x - x^*\|$.

Note that Assumptions A1) and A2) are not the weakest possible assumptions on the function L for convergence of the SPSA or RDKW algorithms; for example, a weaker Lyapunov-type of condition is considered by Chen *et al.* in [2] for convergence of the SPSA algorithm. Since our main objective is not to obtain convergence results under weaker conditions on L, we adopt the more restrictive assumptions [A1) and A2)] to avoid unnecessary complications that may arise from considering the more general L. Throughout this paper, $\{c_n\}$ is a positive scalar sequence with $\lim_{n\to\infty} c_n = 0$.

II. Convergence of Robbins–Monro Algorithms

We rely mainly on the following convergence theorem from [14] and [15] to derive conditions on the perturbations and noise.

Theorem 1: Consider the stochastic approximation algorithm

$$x_{n+1} = x_n - a_n f(x_n) + a_n e_n + a_n b_n \tag{1}$$

where $\{x_n\}$, $\{e_n\}$, and $\{b_n\}$ are sequences on R^p, f: $R^p \to R^p$ satisfies Assumption A2), and $\{a_n\}$ is a sequence of positive real numbers satisfying $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, and $\lim_{n\to\infty} b_n = 0$. Suppose that the sequence $\{f(x_n)\}$ is bounded. Then, for any x_1 in R^p, $\{x_n\}$ converges to x^* if and only if $\{e_n\}$ satisfies any of the following conditions.

C1)

$$\lim_{n\to\infty} \left(\sup_{n \leq k \leq m(n,T)} \left\| \sum_{i=n}^{k} a_i e_i \right\| \right) = 0$$

for some $T > 0$, where $m(n,T) \triangleq \max\{k : a_n + \cdots + a_k \leq T\}$.

C2)

$$\lim_{T\to 0} \frac{1}{T} \limsup_{n\to\infty} \left(\sup_{n \leq k \leq m(n,T)} \left\| \sum_{i=n}^{k} a_i e_i \right\| \right) = 0.$$

C3) For any $\alpha, \beta > 0$, and any infinite sequence of nonoverlapping intervals $\{I_k\}$ on N there exists $K \in N$ such that for all $k \geq K$

$$\left\| \sum_{n \in I_k} a_n e_n \right\| < \alpha \sum_{n \in I_k} a_n + \beta.$$

C4) There exist sequences $\{f_n\}$ and $\{g_n\}$ with $e_n = f_n + g_n$ for all n such that

$$\sum_{k=1}^{n} a_k f_k \text{ converges and } \lim_{n\to\infty} g_n = 0.$$

C5) The weighted average $\{\bar{e}_n\}$ of the sequence $\{e_n\}$ defined by

$$\bar{e}_n = \frac{1}{\beta_n} \sum_{k=1}^{n} \gamma_k e_k$$

converges to zero, where

$$\beta_n = \begin{cases} 1, & n = 1 \\ \prod_{k=2}^{n} \frac{1}{1-a_k}, & \text{otherwise} \end{cases}$$
$$\gamma_n = a_n \beta_n.$$

Proof: See [14] for a proof for conditions C1)–C4) and [15] for a proof for condition C5). □

Theorem 1 provides five equivalence necessary and sufficient noise conditions, conditions C1)–C5), for convergence of the standard Robbins–Monro algorithm (1). Note that the assumption that $\{f(x_n)\}$

is bounded can be relaxed by incorporating the projection scheme into the algorithm as in [3]. Since our objective is to investigate the effect of the "random" directions on the algorithm, rather than to weaken the convergence conditions, the form of the convergence result in Theorem 1 suffices for our purpose. In the next section, we will use Theorem 1 to establish the convergence of two stochastic approximation algorithms with randomized directions by writing them into the form of (1).

III. Algorithms with Randomized Directions

In this section, we study two variants of the KW algorithms with randomized directions, including the SPSA and the RDKW algorithms. In contrast to the standard KW algorithm, which moves along an approximation of the gradient at each iteration, these algorithms move along a sequence of randomized directions. Moreover, these algorithms use only two measurements per iteration to estimate the associated directional derivative as in the case of the RDKW algorithm, or some related quantity as in the case of the SPSA algorithm.

We define the randomized directions as a sequence of vectors $\{d_n\}$ on R^p. We denote the ith component of d_n by d_{ni}. Except for Propositions 1 and 2, the sequence $\{d_n\}$ is assumed to be an arbitrary deterministic sequence. The main goal of this section is to establish a deterministic characterization of $\{d_n\}$ that guarantees convergence of the algorithms under reasonable assumptions. Note that we use the same notation d_n to represent the random directions for both the SPSA and RDKW algorithms to elucidate the similarity between them. This does not imply that the same direction sequence should be applied to both algorithms. In general, different requirements on $\{d_n\}$ are needed for convergence of these two algorithms, as illustrated in Theorems 2 and 3.

A. SPSA Algorithm

Although the convergence of Spall's algorithm has been established in [11], it is not clear (at least intuitively) why random perturbations used in the algorithm would result in faster convergence. In both Spall's [11] and Chen's [2] results, conditions on random perturbations for convergence are stated in probabilistic settings. These stochastic conditions provide little insight into the essential properties of perturbations that contribute to the convergence and efficiency of the SPSA algorithm. In this section, we develop a deterministic framework for the analysis of the SPSA algorithm. We present five equivalent deterministic necessary and sufficient conditions on both the perturbation and noise for convergence of the SPSA algorithm, based on Theorem 1. We believe that our sample-path characterization sheds some light on what makes the SPSA algorithm effective.

We now describe a version of the SPSA algorithm. We define a sequence of vectors $\{r_n\}$, related to $\{d_n\}$, by

$$r_n = \left[\frac{1}{d_{n1}}, \cdots, \frac{1}{d_{np}}\right].$$

The SPSA algorithm is described by

$$x_{n+1} = x_n - a_n \frac{y_n^+ - y_n^-}{2c_n} d_n \tag{2}$$

where y_n^+ and y_n^- are noisy measurements of the function L at perturbed points, defined by

$$y_n^+ = L(x_n + c_n r_n) + e_n^+$$
$$y_n^- = L(x_n - c_n r_n) + e_n^-$$

with additive noise e_n^+ and e_n^-, respectively. For convenience, we write

$$f^r(x_n) = \frac{L(x_n + c_n r_n) - L(x_n - c_n r_n)}{2c_n} \tag{3}$$

as an approximation to the directional derivative along the direction r_n, $r_n^T f(x_n)$. To analyze the algorithm, we rewrite (2) into the standard form of the Robbins–Monro algorithm (1)

$$x_{n+1} = x_n - a_n f^r(x_n) d_n + a_n \frac{e_n}{2c_n} d_n \tag{4}$$

$$\begin{aligned} &= x_n - a_n \big(r_n^T f(x_n) - b_n\big) d_n + a_n \frac{e_n}{2c_n} d_n \\ &= x_n - a_n f(x_n) + a_n b_n d_n + a_n \frac{e_n}{2c_n} d_n \\ &\quad - a_n \big(d_n r_n^T - I\big) f(x_n) \end{aligned} \tag{5}$$

by defining

$$\begin{aligned} b_n &= r_n^T f(x_n) - f^r(x_n) \\ e_n &= e_n^- - e_n^+. \end{aligned} \tag{6}$$

The sequence $\{b_n\}$ represents the bias in the directional derivative approximation. The effective noise for the algorithm is the scaled difference between two measurement noise values, $\frac{e_n}{c_n} d_n$. We can apply the result in Theorem 1 to establish the convergence of the SPSA algorithm (2). We first prove that the bias sequence $\{b_n\}$ converges to zero.

Lemma 1: Suppose that L: $R^p \to R$ satisfies Assumption A1), $\{c_n\}$ converges to zero, and $\{r_n\}$ is bounded. Then the sequence $\{b_n\}$ defined by (6) converges to zero.

Proof: By the Mean Value theorem

$$\begin{aligned} b_n &= r_n^T f(x_n) - \frac{L(x_n + c_n r_n) - L(x_n - c_n r_n)}{2c_n} \\ &= r_n^T [f(x_n) - f(x_n + (2\lambda_n - 1) c_n r_n)] \end{aligned}$$

where $0 \le \lambda_n \le 1$ for all $n \in N$. Let $\epsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \frac{\epsilon}{\sup_n \|r_n\|}$. Furthermore, by the convergence of $\{c_n\}$ there exists $N \in N$ such that $\|(2\lambda_n - 1) c_n r_n\| < \delta$ for all $n \ge N$. Hence, for all $n \ge N$

$$\begin{aligned} |b_n| &\le \|r_n\| \|f(x_n) - f(x_n + (2\lambda_n - 1) c_n r_n)\| \\ &< \sup_n \|r_n\| \frac{\epsilon}{\sup_n \|r_n\|} = \epsilon. \end{aligned}$$

Therefore $\{b_n\}$ converges to zero. □

Using Theorem 1 and Lemma 1, we establish a necessary and sufficient condition for convergence of the SPSA algorithm in the following theorem.

Theorem 2: Suppose that Assumptions A1 and A2) hold, and $\{r_n\}$, $\{d_n\}$, and $\{f(x_n)\}$ are bounded. Then, $\{x_n\}$ defined by (2) converges to x^* if and only if the sequences $\{(d_n r_n^T - I) f(x_n)\}$ and $\{\frac{e_n}{2c_n} d_n\}$ satisfy conditions C1)–C5).

Proof ($\Rightarrow$)*:* Suppose that $\{x_n\}$ converges to x^*. Then $\{f(x_n)\}$ converges to $f(x^*) = 0$ by the continuity of f. Since $\{r_n\}$ and $\{d_n\}$ are bounded, $\|d_n r_n^T - I\|$ is bounded and $\{(d_n r_n^T - I) f(x_n)\}$ converges to zero. Thus $\{(d_n r_n^T - I) f(x_n)\}$ satisfies conditions C1)–C5). By Theorem 1, $\{(d_n r_n^T - I) f(x_n) - \frac{e_n}{2c_n} d_n\}$ satisfies C1)–C5). Therefore $\{\frac{e_n}{2c_n} d_n\}$ satisfies conditions C1)–C5).

($\Leftarrow$)*:* This follows directly from Theorem 1 and Lemma 1. □

Theorem 2 provides the tightest possible conditions on both the randomized direction sequence $\{d_n\}$ and noise sequence $\{e_n\}$. Note that the condition on d_n is "coupled" with the function values $f(x_n)$. This special form of coupling suggests an adaptive design scheme for d_n based on the estimate of $f(x_n)$. However, this idea may be difficult to carry out due to the special structure of the matrix

$$d_n r_n^T - I = \begin{bmatrix} 0 & \frac{d_{n1}}{d_{n2}} & \cdots & \frac{d_{n1}}{d_{np}} \\ \frac{d_{n2}}{d_{n1}} & 0 & \cdots & \frac{d_{n2}}{d_{np}} \\ \vdots & \vdots & & \vdots \\ \frac{d_{np}}{d_{n1}} & \frac{d_{np}}{d_{n2}} & \cdots & 0 \end{bmatrix}. \tag{7}$$

We can see that it is difficult to scale the elements in the matrix according to $\{f(x_n)\}$. One solution to this is to establish probabilistic sufficient conditions on the perturbation to guarantee that the deterministic condition in Theorem 1 holds almost surely, as in [2] and [11]. We present a general sufficient condition based on the martingale convergence theorem. In the following proposition, we assume that $\{d_n\}$ and $\{e_n\}$ are random sequences.

Proposition 1: Let $\mathcal{F}_n$ be the σ-algebra generated by $\{d_k\}_{k=1,\cdots,n}$ and $\{e_k\}_{k=1,\cdots,n}$. Assume that $\sum_{n=1}^{\infty} a_n^q < \infty$ for some $q > 1$, and $E(\frac{d_{ni}}{d_{nj}} \mid \mathcal{F}_{n-1}) = 0$ for $i \neq j$. Suppose that $\{d_n\}$, $\{r_n\}$, and $\{f(x_n)\}$ are bounded. Then, $\{(d_n r_n^T - I)f(x_n)\}$ satisfies noise conditions C1)–C5) almost surely.

Proof: Let $z_n = a_n(d_n r_n^T - I)f(x_n)$, $z_n = [z_{n1}, \cdots, z_{np}]^T$. Since

$$E(z_n \mid \mathcal{F}_{n-1}) = E\left[a_n\left(d_n r_n^T - I\right)f(x_n) \mid \mathcal{F}_{n-1}\right] = 0$$

$\{\sum_{k=1}^{n} z_k\}$ is a martingale. Furthermore

$$E(|z_{ni}|^q) < \infty$$

for all $i \leq p$ by the boundedness of $\{d_n\}$, $\{r_n\}$, $\{f(x_n)\}$, and $\sum_{n=1}^{\infty} a_n^p$. Hence by the L^q convergence theorem for martingales [5, eq. (4.4), p. 217], the sequence $\{\sum_{k=1}^{n} z_k\}$ converges almost surely. Therefore, $\{(d_n r_n^T - I)f(x_n)\}$ satisfies condition C4) and hence the noise conditions C1)–C5). □

In [2] and [11], d_n is assumed to be a vector of p mutually independent random variables independent of $\mathcal{F}_{n-1}$. Under this assumption, it is clear that the condition in Proposition 1 can be satisfied by assuming either $E(d_{ni}) = 0$, as in [2], or $E(\frac{1}{d_{ni}}) = 0$, as in [11].

B. RDKW Algorithms

The RDKW algorithm with random directions of different distributions has been studied by several authors; see, for example, [6], [8], and [12]. In this section, we study the convergence of the RDKW algorithm for a general direction sequence $\{d_n\}$. We establish deterministic necessary and sufficient conditions on both the direction sequence $\{d_n\}$ and the noise sequence for the convergence of the RDKW algorithm (Theorem 3). We compare and contrast these conditions to those of Theorem 2.

The RDKW algorithm can be described by

$$x_{n+1} = x_n - a_n \frac{y_n^+ - y_n^-}{2c_n} d_n \tag{8}$$

where

$$\begin{aligned} y_n^+ &= L(x_n + c_n d_n) + e_n^+ \\ y_n^- &= L(x_n - c_n d_n) + e_n^-. \end{aligned}$$

As in the case of the SPSA algorithm, we can rewrite the algorithm (8) into a Robbins–Monro algorithm

$$\begin{aligned} x_{n+1} &= x_n - a_n f^d(x_n) d_n + a_n \frac{e_n}{2c_n} d_n \\ &= x_n - a_n \rho^2 f(x_n) + a_n b_n d_n + a_n \frac{e_n}{2c_n} d_n \\ &\quad - a_n\left(d_n d_n^T - \rho^2 I\right) f(x_n) \end{aligned} \tag{9}$$

by defining

$$\begin{aligned} b_n &= d_n^T f(x_n) - f^d(x_n) \\ e_n &= e_n^- - e_n^+ \end{aligned} \tag{10}$$

where $\rho > 0$ is an arbitrary constant real number. Comparing (9) with (4), we can see that the RDKW algorithm differs from the SPSA algorithm only in the direction along which the directional derivative is estimated.

Following the same arguments as in the proofs of Lemma 1 and Theorem 2, we can show that the sequence $\{b_n\}$ defined by (10) converges to zero and the following theorem holds.

Theorem 3: Suppose that Assumptions A1) and A2) hold, and $\{d_n\}$ and $\{f(x_n)\}$ are bounded. Then, $\{x_n\}$ defined by (8) converges to x^* if and only if the sequences $\{(d_n d_n^T - \rho^2 I)f(x_n)\}$ and $\{\frac{e_n}{2c_n} d_n\}$ satisfy noise conditions C1)–C5).

Comparing the above conditions with those of Theorem 2, we again notice a coupling between $\{d_n\}$ and $\{f(x_n)\}$. Similar to the case of the SPSA algorithm, it may be difficult to design $\{d_n\}$ based on $\{f(x_n)\}$ to satisfy the condition for $\{(d_n d_n^T - \rho^2 I)f(x_n)\}$ above, due to the structure of the matrix

$$d_n d_n^T - \rho I = \begin{bmatrix} (d_{n1})^2 - \rho^2 & d_{n1}d_{n2} & \cdots & d_{n1}d_{np} \\ d_{n2}d_{n1} & (d_{n2})^2 - \rho^2 & \cdots & d_{n2}d_{np} \\ \vdots & \vdots & & \vdots \\ d_{np}d_{n1} & d_{np}d_{n2} & \cdots & (d_{np})^2 - \rho^2 \end{bmatrix}. \tag{11}$$

Although we are allowed to choose smaller d_{ni} (the same is not true for SPSA since $\frac{1}{d_{ni}}$ needs to be bounded), the diagonal terms always give a "weight" around the quantity ρ^2. Furthermore, if we try to choose $\rho^2 = \rho_n^2$ such that $\rho_n^2 \approx (d_{ni})^2$ and let $\rho_n \to 0$, the resulting sequence of functions $\{\rho_n^2 f(x_n)\}$ in (9) may not satisfy Assumptions A1) and A2) and the algorithm may not converge. However, similar to the case for the SPSA algorithm, we can derive the following sufficient probabilistic condition on d_n. As before, we assume below that $\{d_n\}$ and $\{e_n\}$ are random sequences.

Proposition 2: Let $\mathcal{F}_n$ be the σ-algebra generated by $\{d_k\}_{k=1,\cdots,n}$ and $\{e_k\}_{k=1,\cdots,n}$. Assume that $\sum_{n=1}^{\infty} a_n^q < \infty$ for some $q > 1$, and $E(d_n d_n^T \mid \mathcal{F}_{n-1}) = \rho^2 I$. Suppose that $\{d_n\}$ and $\{f(x_n)\}$ are bounded. Then $\{(d_n d_n^T - \rho^2 I)f(x_n)\}$ satisfies conditions C1)–C5) almost surely.

IV. Conclusion

As pointed out in Section III, the SPSA and RDKW algorithms are very similar in form. In fact, under a probabilistic framework, we can show that these two algorithms have the same asymptotic performance under optimized conditions. In [10], Sadegh and Spall show that the random direction sequence $\{d_n\}$ with Bernoulli distribution for each component is asymptotically optimal for the SPSA algorithm. Following the same approach, we can show that the Bernoulli distribution is also optimal for the RDKW algorithm, based on the asymptotic distribution established by Chin in [4] for the RDKW algorithm. These two algorithms are clearly identical in this case and hence exhibit the same performance (see [13, Sec. 5.4] for a proof).

Although the probabilistic approach used in [4] and [11] is useful in assessing the asymptotic performance of stochastic approximation algorithms, it provides little insight into the essential properties of randomized directions that contribute to the convergence and efficiency of the SPSA and RDKW algorithms. Furthermore, simulation results suggest that the SPSA algorithm (or the RDKW algorithm) with Bernoulli distributed randomized directions outperforms the standard KW algorithm not only in the average but also along each sample path. This phenomenon certainly is not reflected in the probabilistic analysis of [11] or [4]. Our deterministic analysis does shed some light on what makes the SPSA and the RDKW algorithms effective. Perhaps a complete answer can be obtained by further analyzing the sequence $\{(d_n r_n^T - I)f(x_n)\}$ (or $\{(d_n d_n^T - \rho^2 I)f(x_n)\}$ for the RDKW), which is the difference between the SPSA (or the RDKW) algorithm and the standard KW algorithm. In the case where $d_{ni} = \pm 1$, as in the case of Bernoulli distribution, the matrix $d_n r_n^T - I$ (or $d_n d_n^T - I$) has a unique structure

$$\begin{bmatrix} 0 & \frac{d_{n1}}{d_{n2}} & \cdots & \frac{d_{n1}}{d_{np}} \\ \frac{d_{n2}}{d_{n1}} & 0 & \cdots & \frac{d_{n2}}{d_{np}} \\ \vdots & \vdots & & \vdots \\ \frac{d_{np}}{d_{n1}} & \frac{d_{np}}{d_{n2}} & \cdots & 0 \end{bmatrix}.$$

We conjecture that the effect of the elements in the sequence $\{(d_n r_n^T - I)f(x_n)\}$ will be "averaged out" [in the sense of condition C5)] across the iterations along each sample path, if the directions $\{d_n\}$ are chosen appropriately.

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