Priority Queues

What is a Priority Queue?

1) Stores prioritized key-value pairs

2) Implements insertion

- No notion of storing at particular position

3) Returns elements in priority order

- Order determined by *key*

What's so different?

Stacks and Queues

- Removal order determined by order of inserting

Sequences

- User chooses exact placement when inserting and explicitly chooses removal order

Priority Queue

- Order determined by key
- Key may be part of element data or separate

What's it good for?

Order of returned elements is not FIFO or LIFO (as in queue or stack)

Random access not necessary (as in sequence) or desirable

Examples

- Plane landings managed by air traffic control
- Processes scheduled by CPU
- College admissions process for students
  - What are some of the criteria?

College Admissions Key

Student submits:

- Personal data (geography, is parent alum?, activities?)
- Transcript
- Essays
- Standardized test scores
- Recommendations

Admissions agent:

- Each datum converted to number
- Formula converts to single numeric key

Student selection process

Simple scheme

- Collect applications until due date
- Sort by keys
- Take top *k* students

More realistic

- Prioritize applications as they come in
- Accept some top students ASAP
- Maybe even change data/key as you go

Entry ADT

An **entry** in a priority queue is simply a key-value pair

Priority queues store entries to allow for efficient insertion and removal based on keys

Methods:

- getkey(): returns the key for this entry
- getvalue(): returns the value associated with this entry

As a Java interface:

```
/**
 * Interface for a key-value
 * pair entry
 **/
public interface Entry <K,V> {
    public K getkey();
    public V getvalue();
}
```

Priority Queue ADT

```
public interface PriorityQueue<K extends
  Comparable<? super AnyType>,V>
{
  public int size();
  public boolean isEmpty();
  public Entry<K,V> min() throws
  EmptyPriorityQueueException;
  public Entry<K,V> insert(K key, V value) throws
  InvalidKeyException;
  public Entry<K,V> removeMin() throws
  EmptyPriorityQueueException;
}
```

Implementing PQ with Unsorted Sequence

Each call to insertItem(k, e) uses insertLast( ) to store in Sequence

- $O(1)$ time

Each call to extractMin( ) traverses the entire sequence to find the minimum, then removes element

- $O(n)$ time

Implementing PQ with Sorted Sequence

Each call to insertItem(k, e) traverses sorted sequence to find correct position, then does insert

- O(n) worst case

Each call to extractMin( ) does removeFirst( )

- $O(1)$ time

Implementing PQ with a BST

Each call to insertItem(k, e) does tree insert

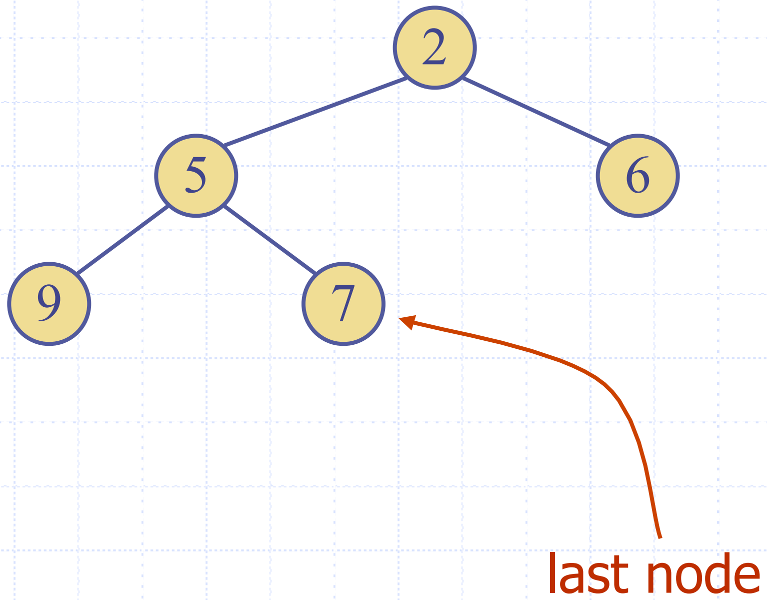
- *O*(log(n)) worst case

Each call to extractMin( ) does delete( )

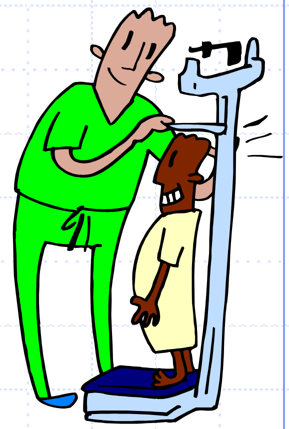
- *O*(*log*(*n*)) time

Heaps

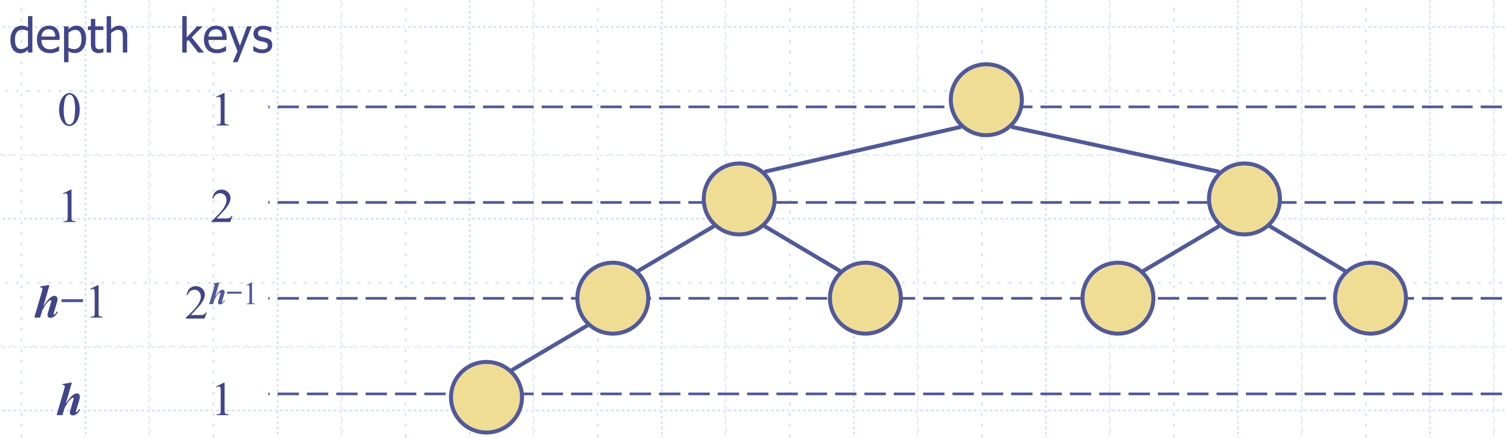
- A heap is a binary tree storing keys at its nodes and satisfying the following properties:
  - Heap-Order: for every internal node v other than the root, $key(v) \geq key(parent(v))$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0, \dots, h - 1$, there are 2^i nodes of depth i
    - at depth $h - 1$, the internal nodes are to the left of the external nodes
- The last node of a heap is the rightmost node of depth h



Height of a Heap



- Theorem: A heap storing n keys has height $O(\log n)$
- Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h - 1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
  - Thus, $n \geq 2^h$, i.e., $h \leq \log n$



Heap

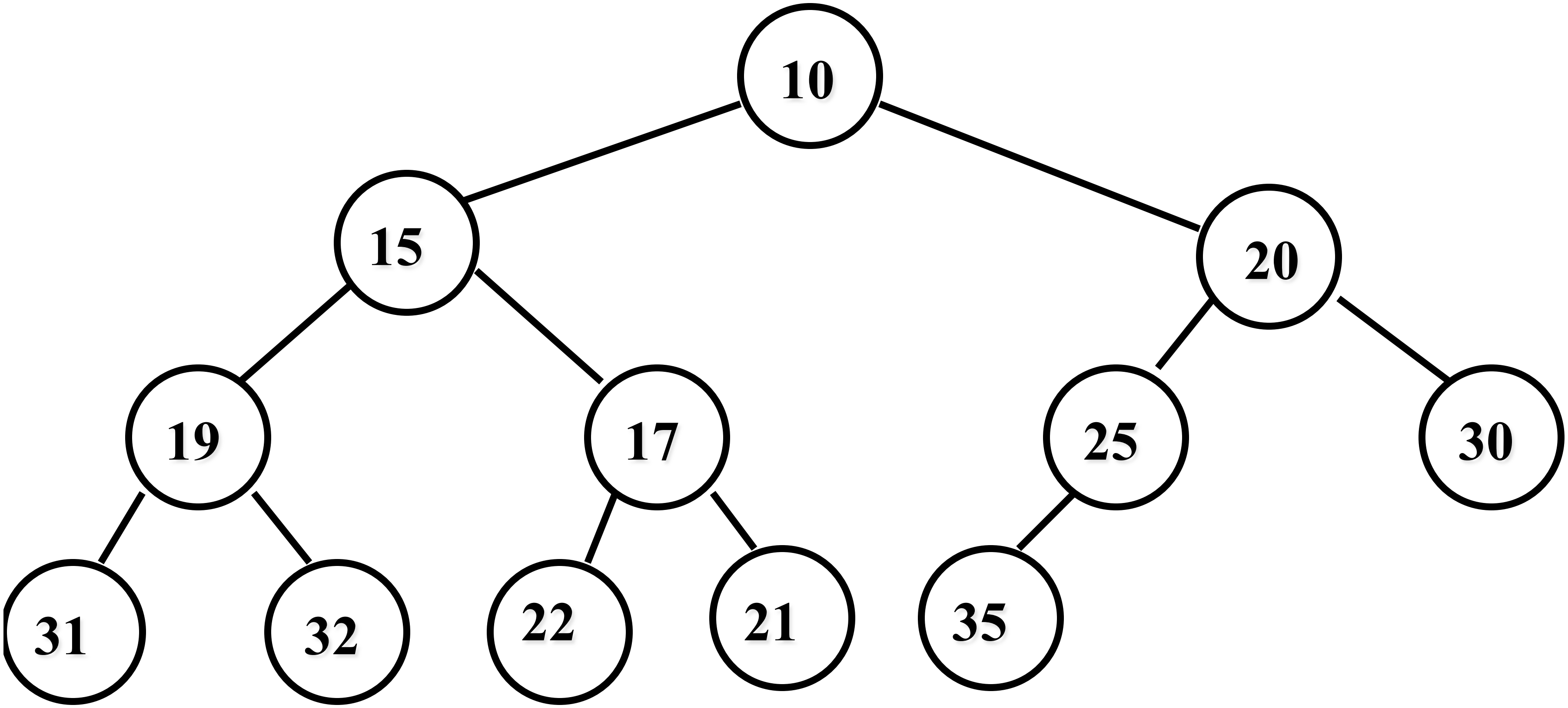
Binary tree-based data structure

- *Complete* in the sense that it fills up levels as completely as possible
- Height of tree is $O(\log n)$

Can be stored using the array representation (just add at the end of the array)

Use extendable arrays to expand and shrink as needed

Heap Example

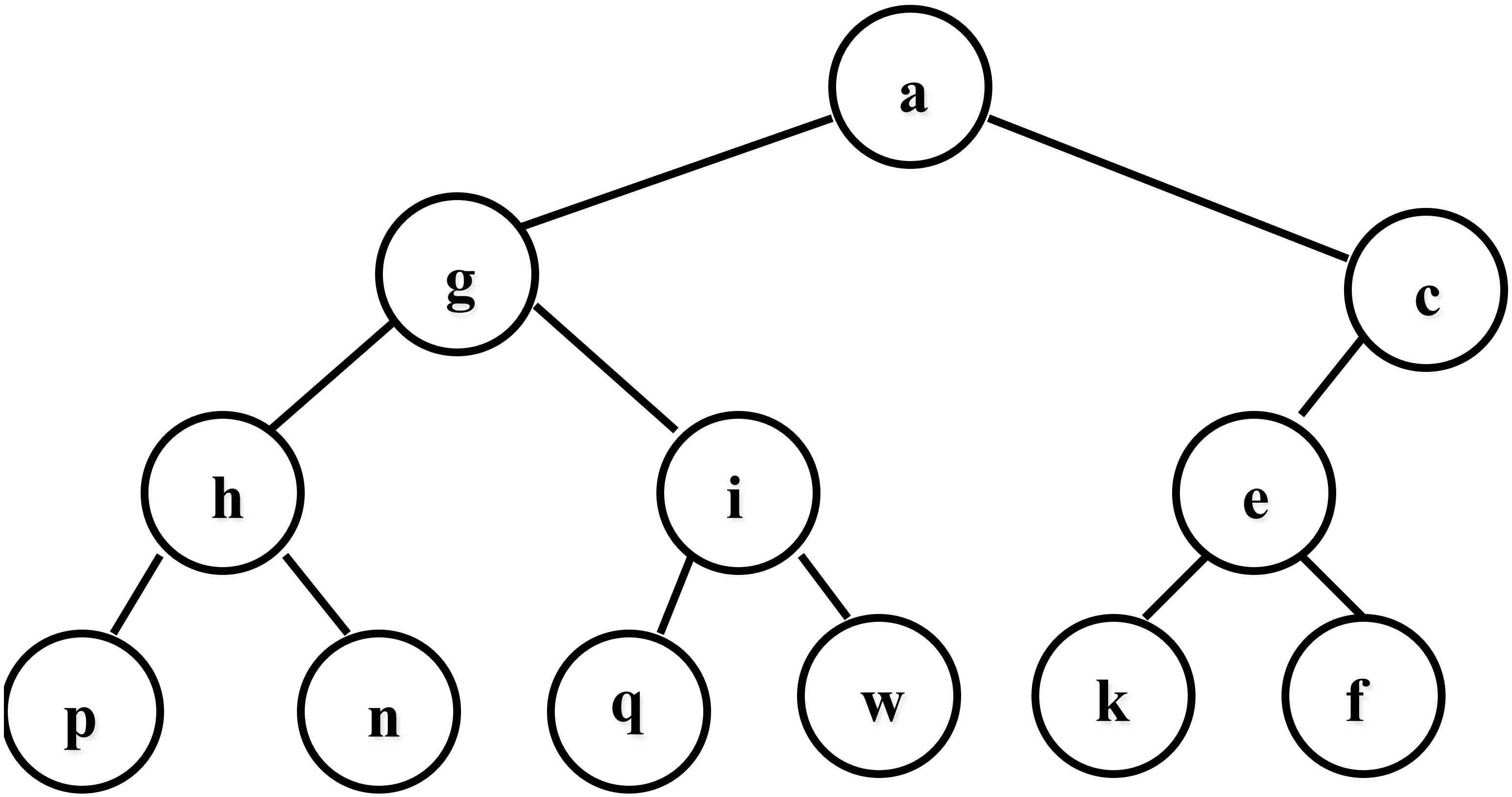


PQ Quiz Show!

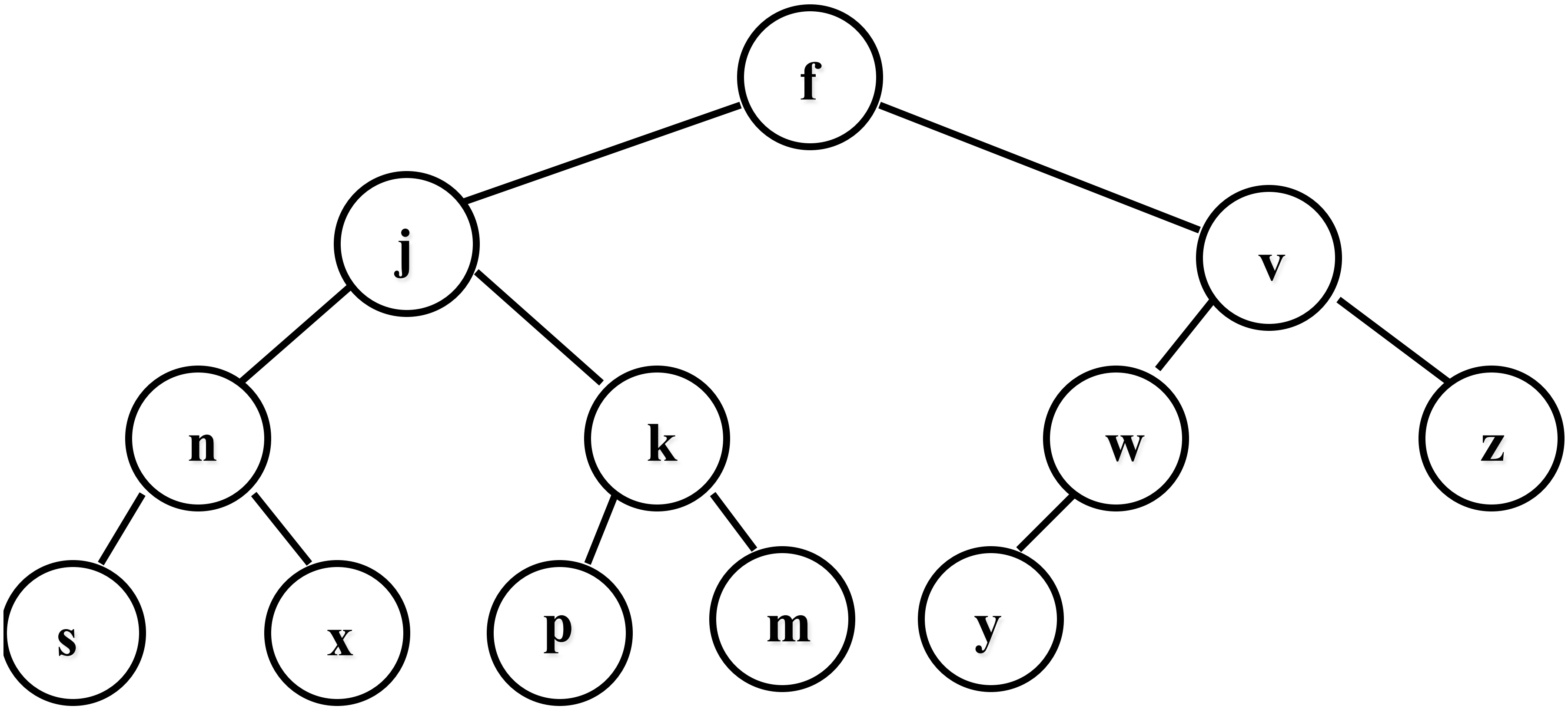
Heap, or Not A Heap?

(no paper necessary)

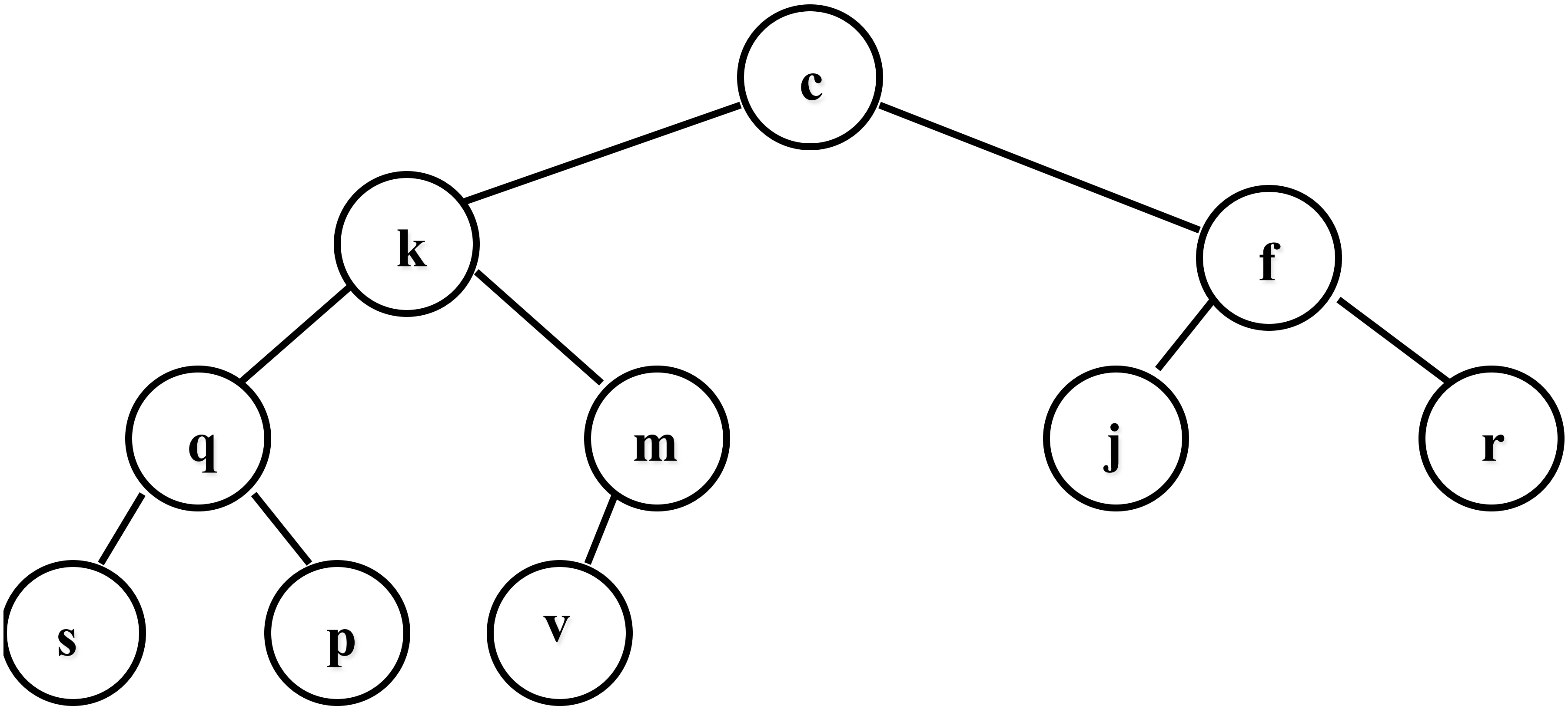
Heap, or Not a Heap?



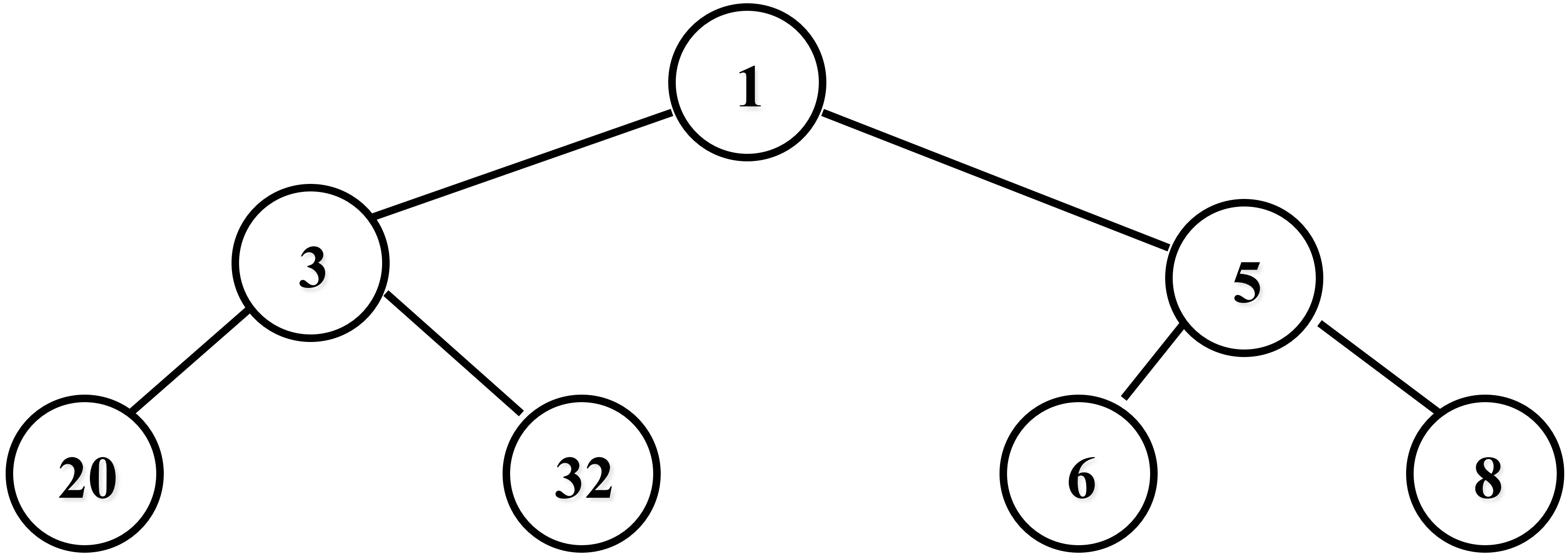
Heap, or Not a Heap?



Heap, or Not a Heap?



Heap, or Not a Heap?



Inserting into Heap

Create new node as "last" element

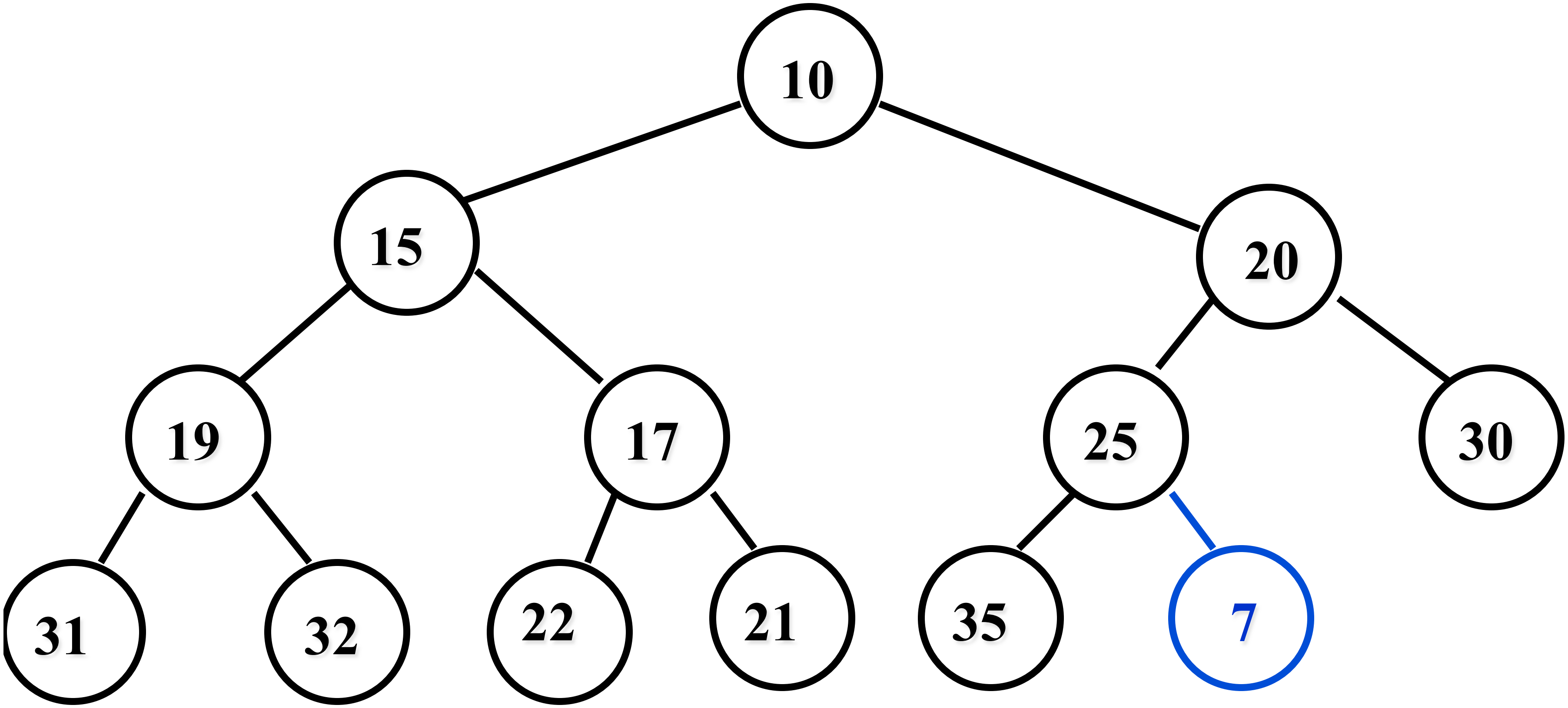
Insert key/element into new node

Bubble node upward until heap property is satisfied

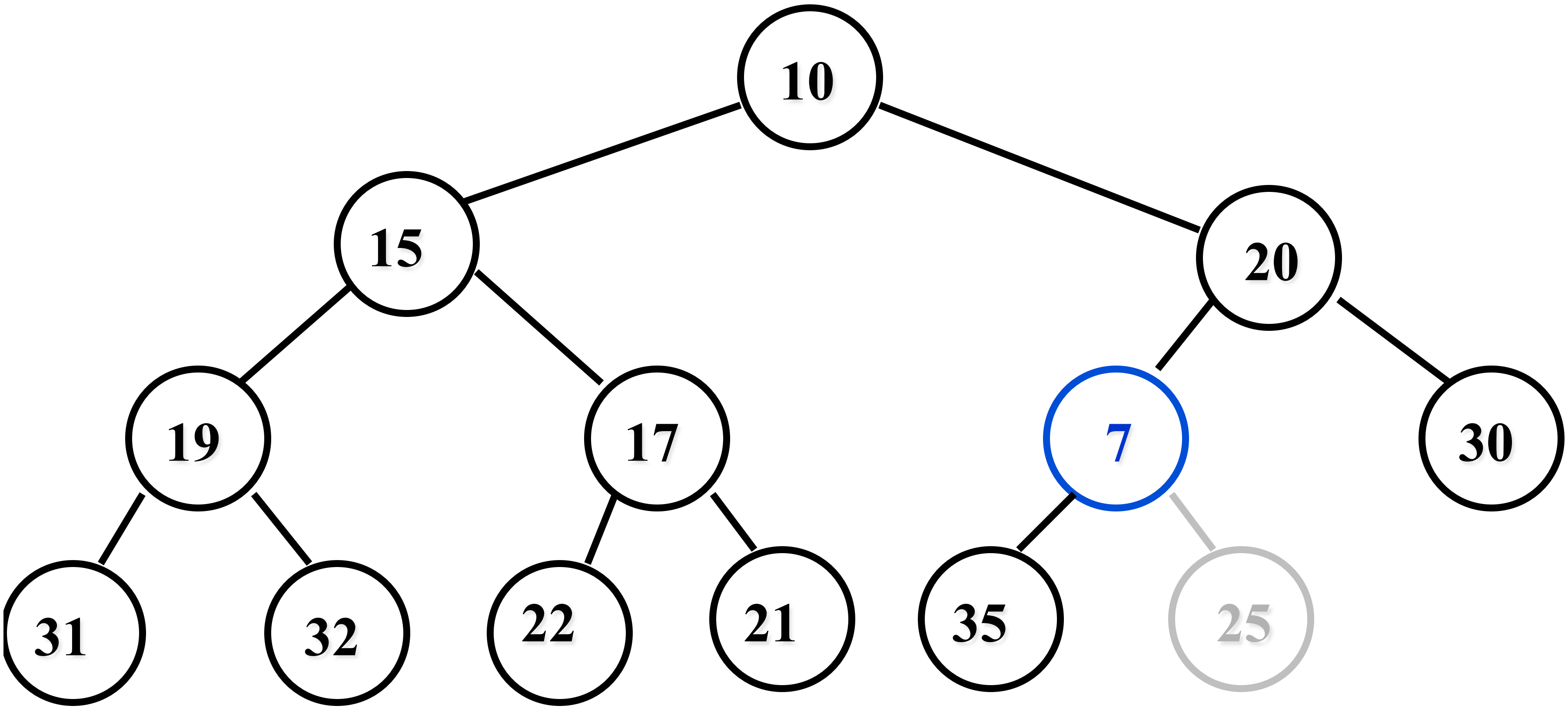
```
while (!isRoot(node) &&
  (node.key < node.parent.key))
      swap(node, parent)
```

(can we do even better?)

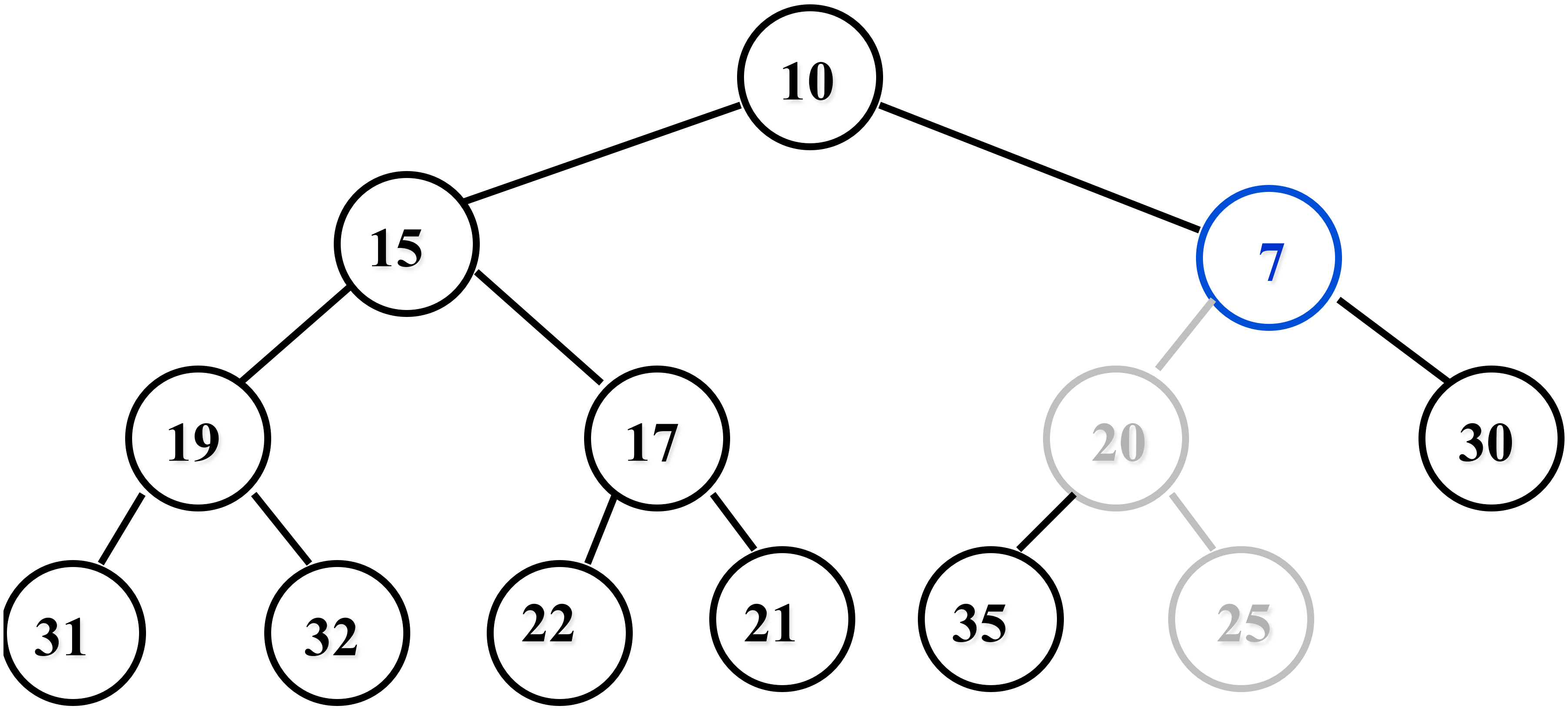
Heap Insert Example



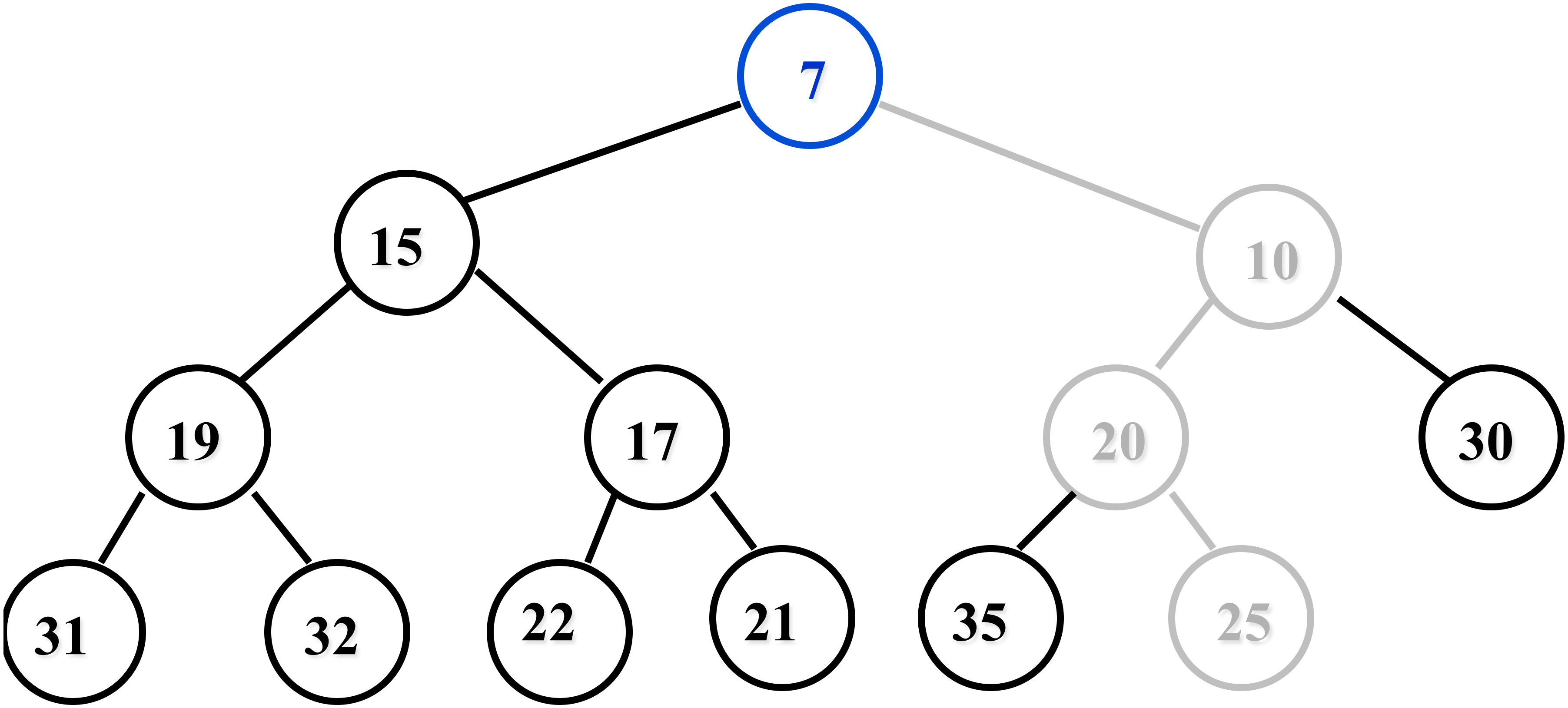
Bubble Upward



Bubble Upward



Bubble Upward



Heap Insert Analysis

New node always inserted at lowest level

Node bubbles upward

- up to root in worst case
- path length to root is $O(\log n)$

Total time for insert is $O(\log n)$

Extracting from Heap

Copy element from root node

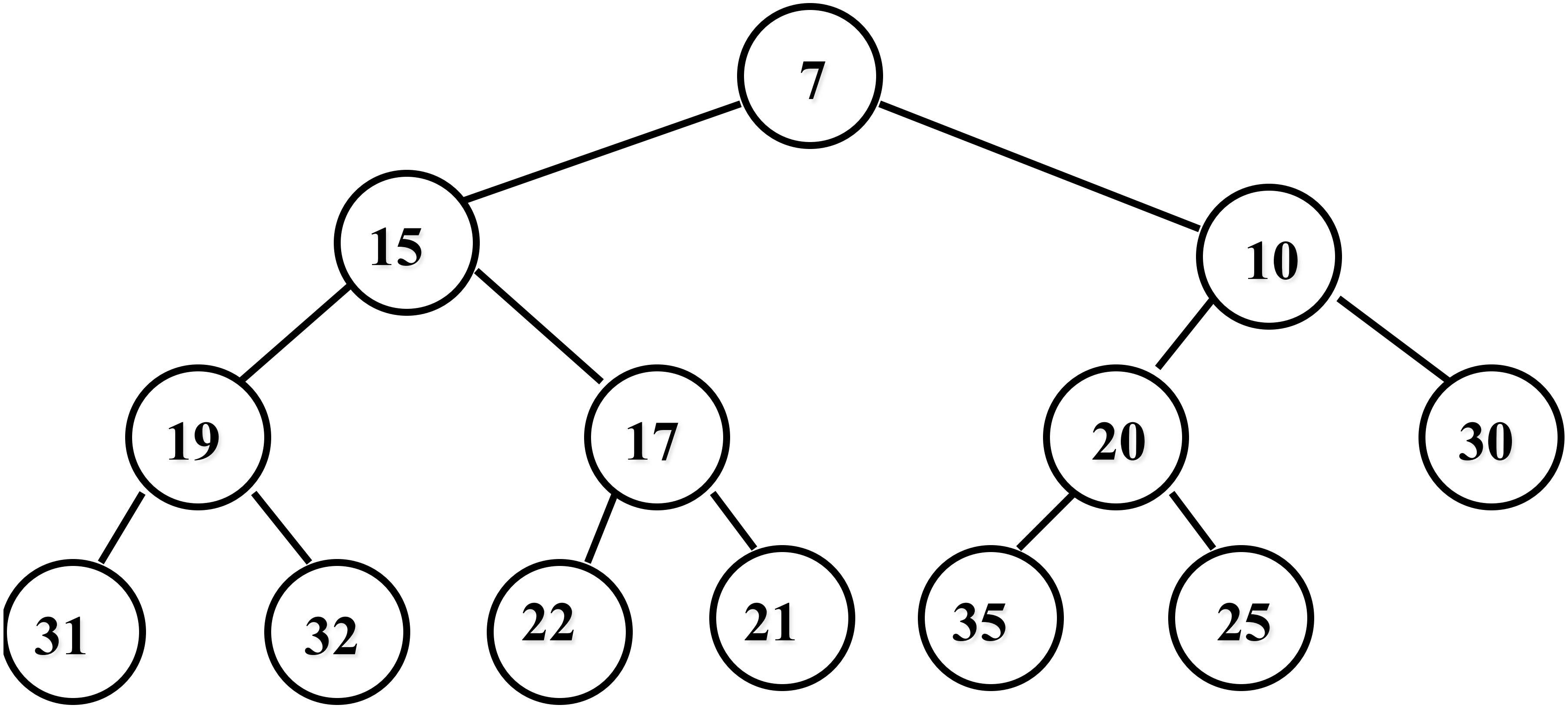
Copy element/key from last node to root node

Delete last node

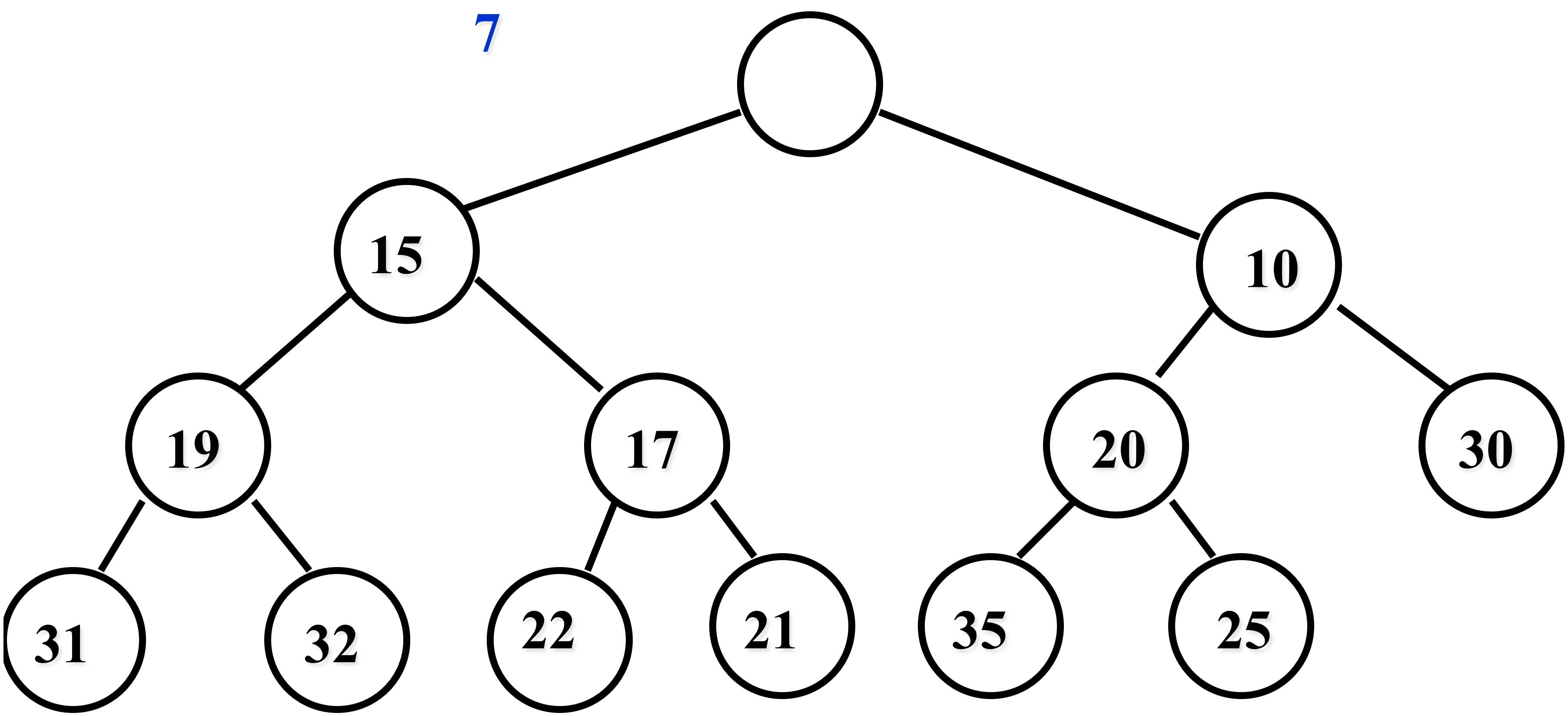
Bubble root node downward until heap property satisfied

```
while (!isExternal(node) &&
        (node.key > node.smallestChild.key))
    swap(node, node.smallestChild)
```

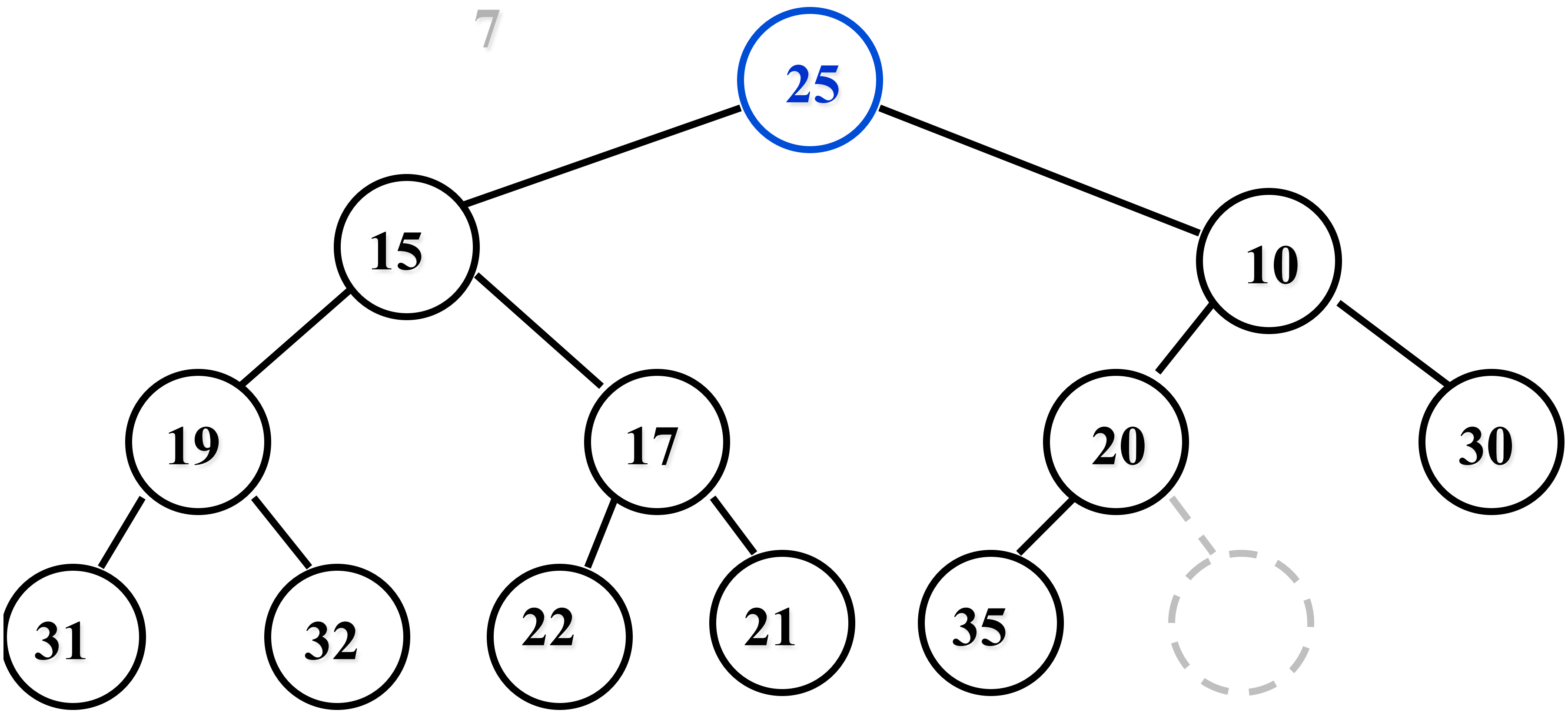
Heap Extract Example



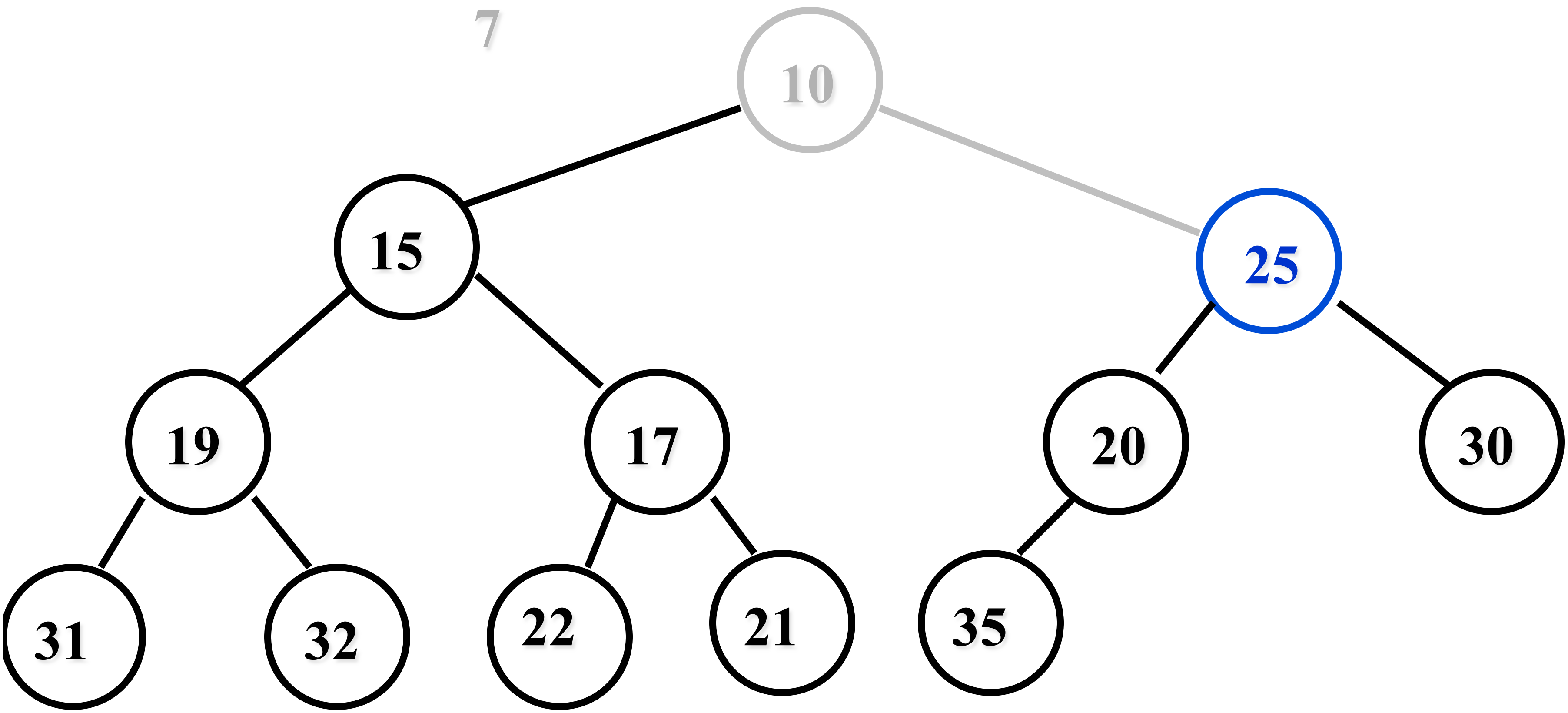
Copy Extracted Element



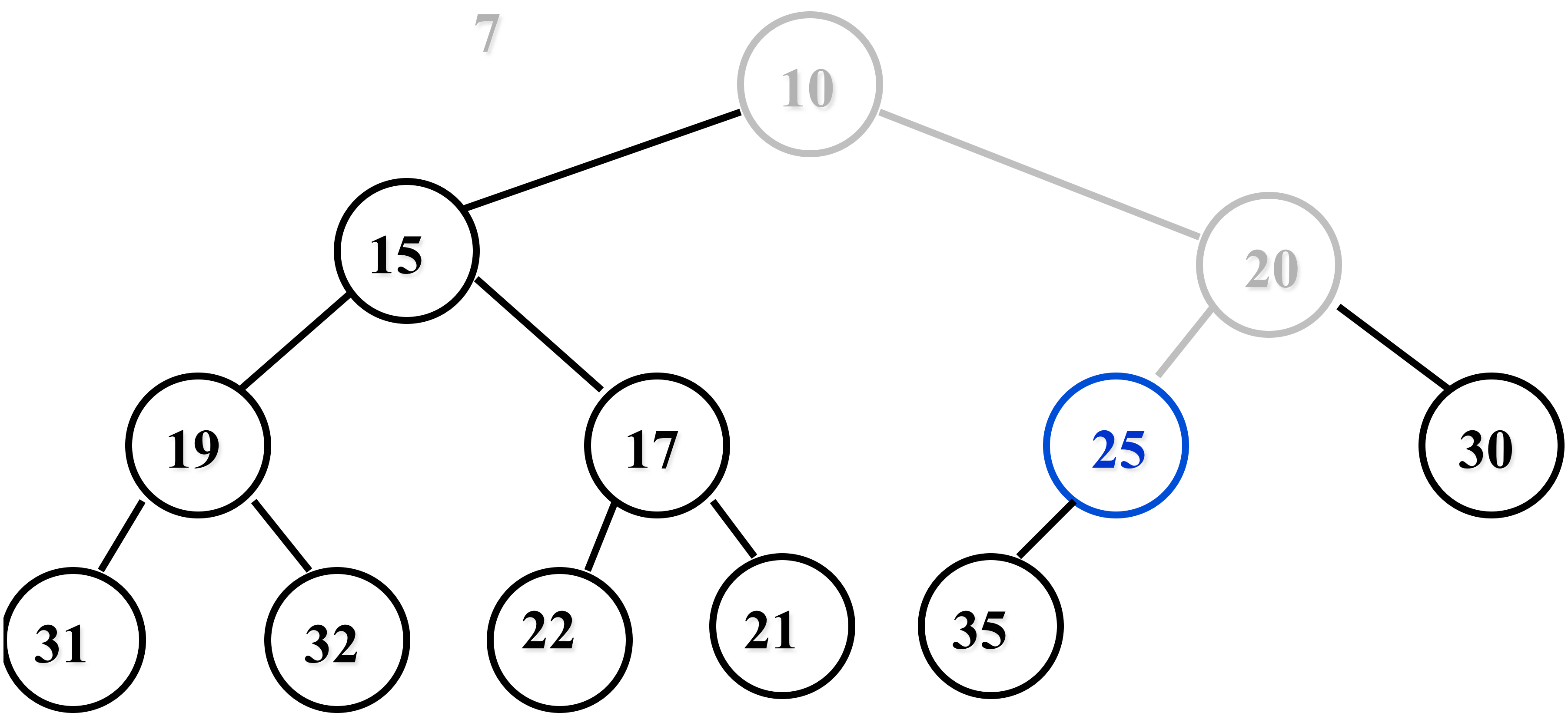
Move last node to root



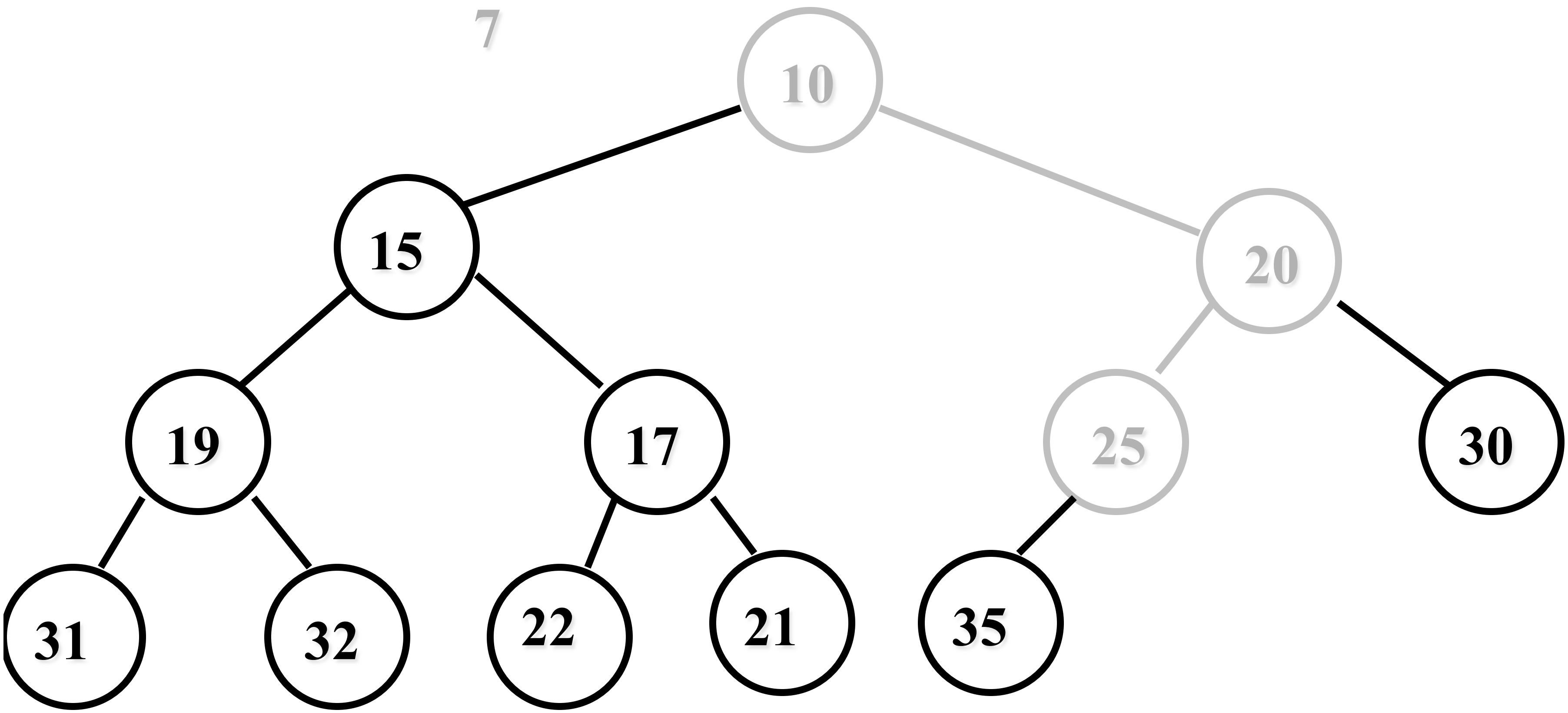
Bubble Downward



Bubble Downward



All done



Heap Extract Analysis

Again, each swap takes constant time

Maximum swaps is path length from root to leaf

→ Total work is $\log n * O(1) = O(\log n)$

Sort Analysis

Heap Sort

```
foreach element, Ei, in S
   PQ.insert(Ei)
while !PQ.empty()
   PQ.extractMin()
```

$O(n)$

$$\sum_{i=0}^{i=n-1} O(\log i)$$

$O(n)$

$$\sum_{i=0}^{i=n-1} O(\log i)$$

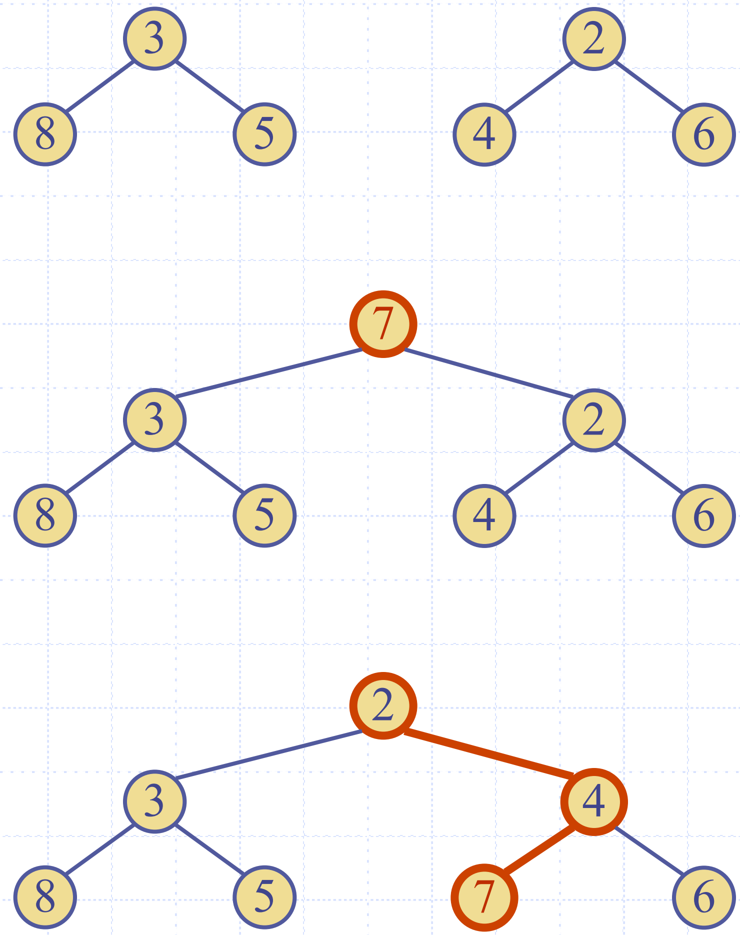
$$O(n) + 2\sum_{i=0}^{i=n-1} O(\log i) < O(n) + 2\sum_{i=0}^{i=n-1} O(\log n)$$

$$= O(n) + 2n * O(\log n) = O(n\log n)$$

(showing $\theta(n\log n)$ is a bit harder)

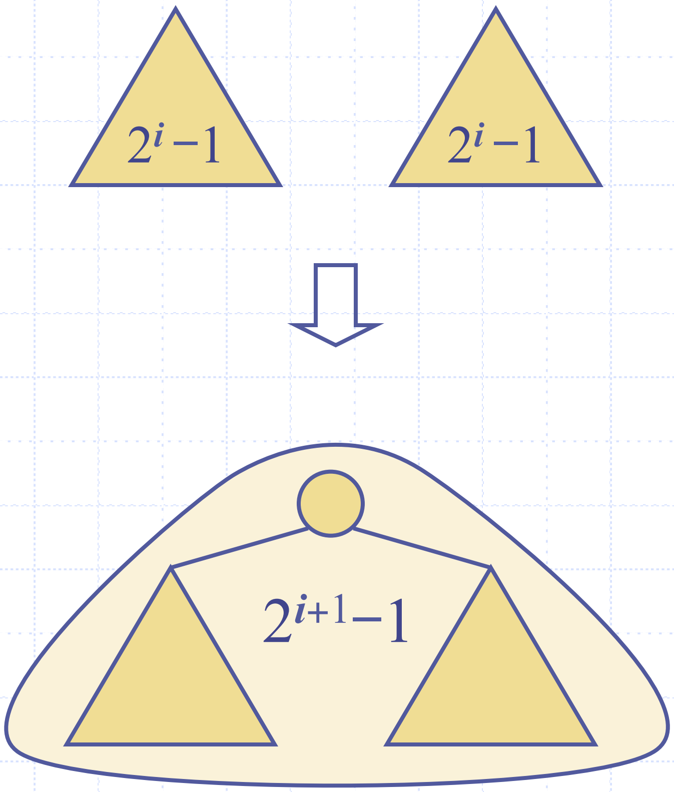
Merging Two Heaps

- We are given two heaps and a key ***k***
- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

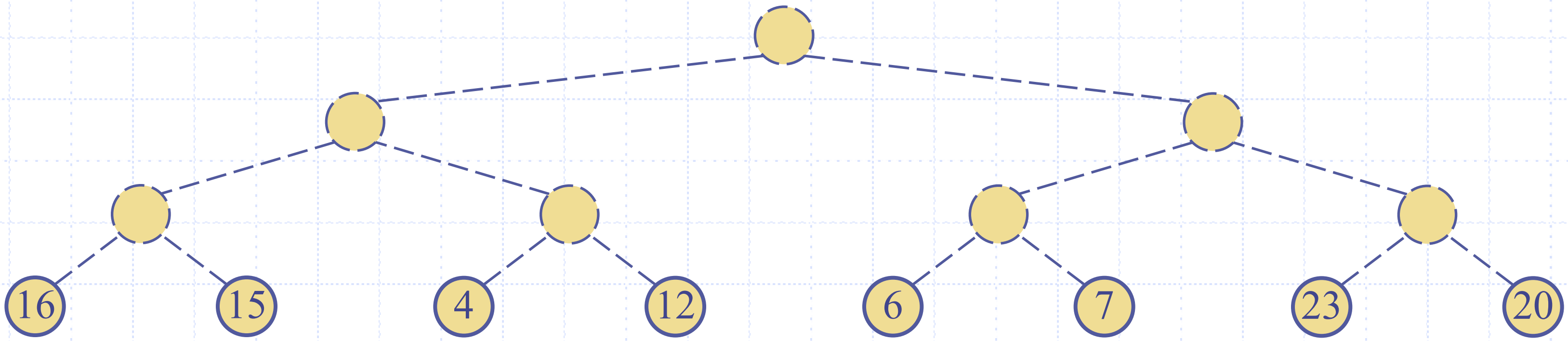


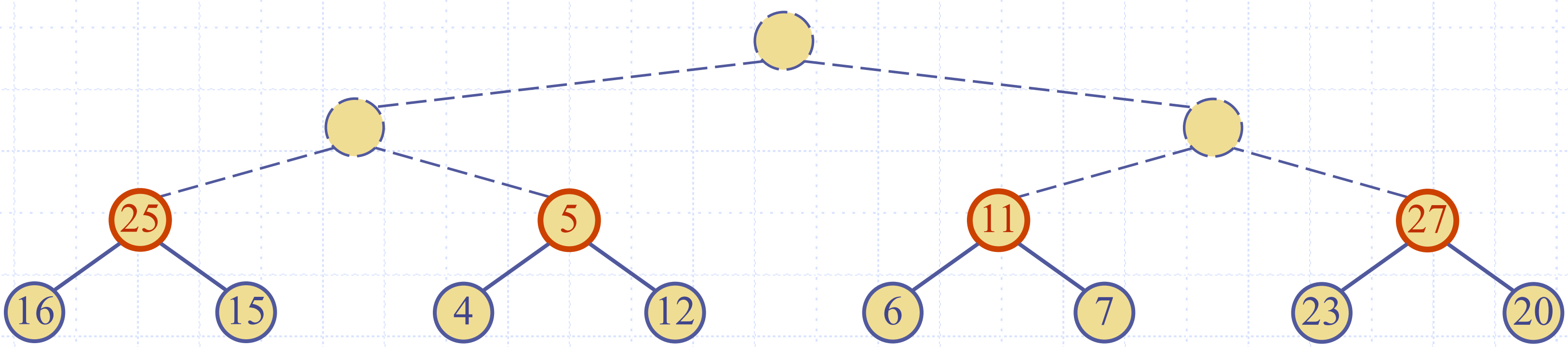
Bottom-up Heap Construction

- We can construct a heap storing ***n*** given keys in using a bottom-up construction with log ***n*** phases
- In phase ***i***, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys

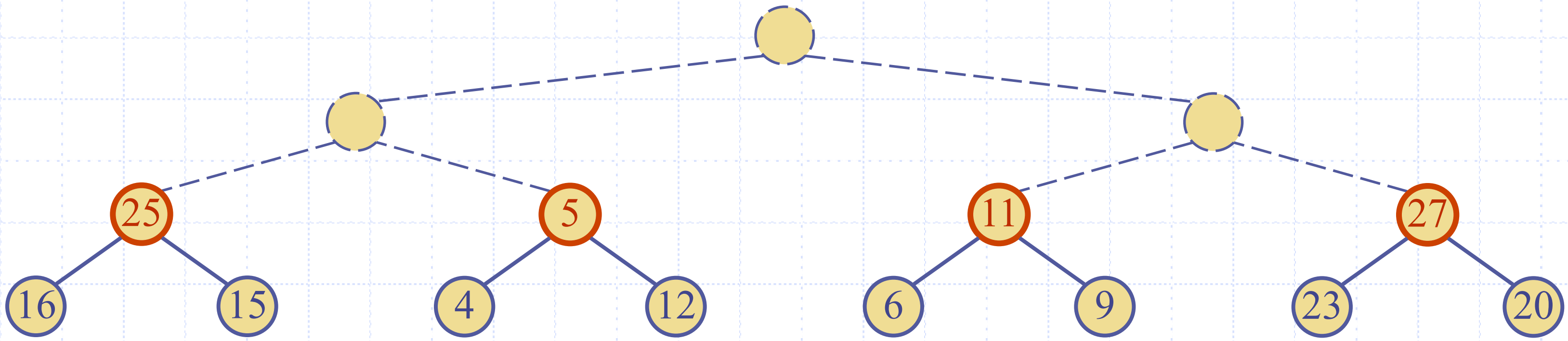


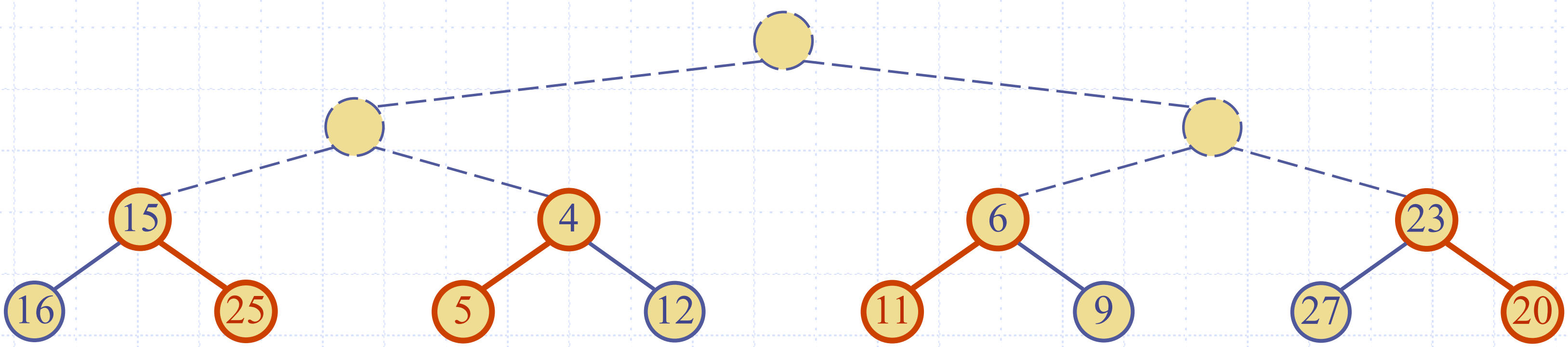
Example



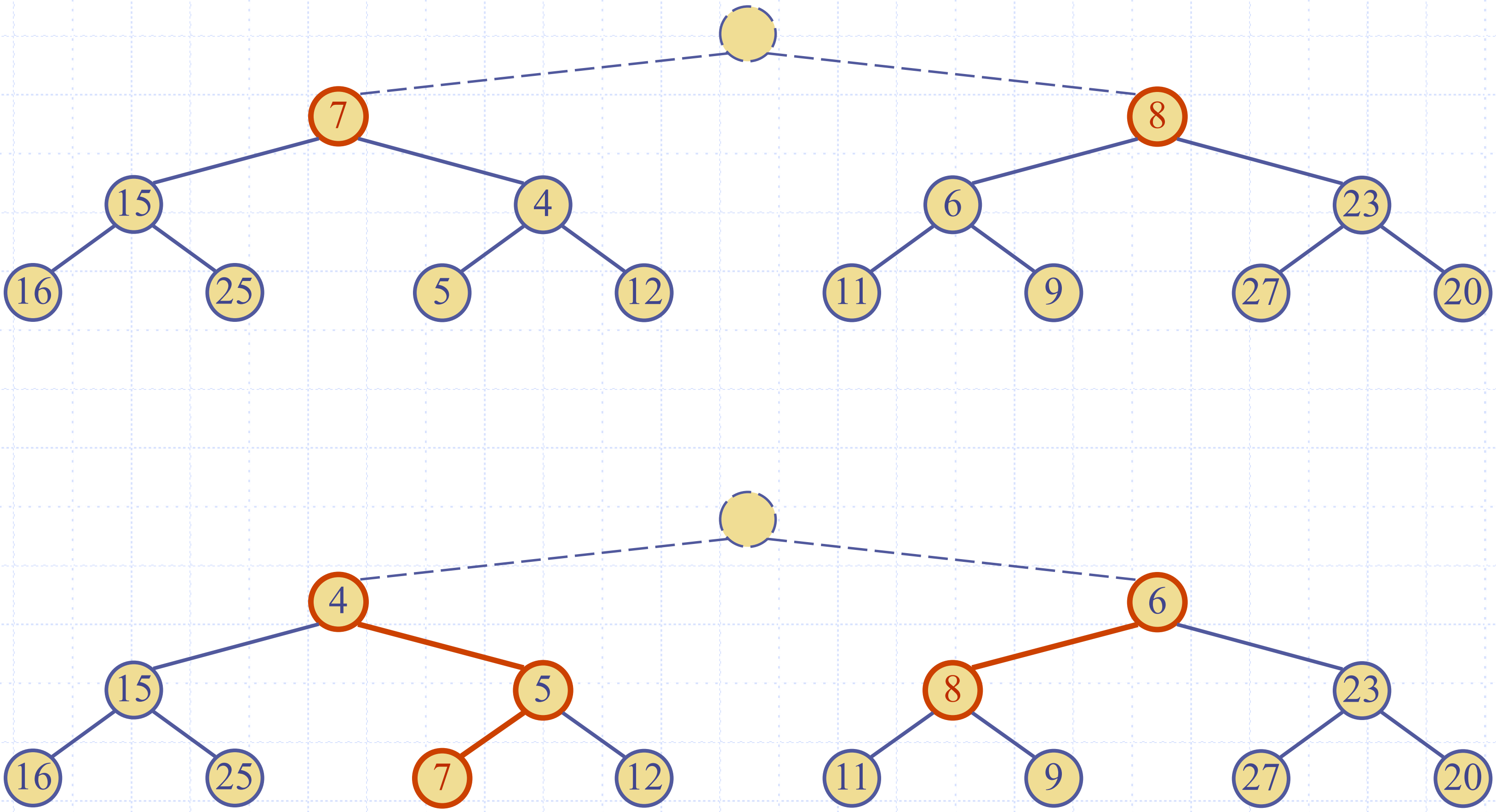


Example (contd.)

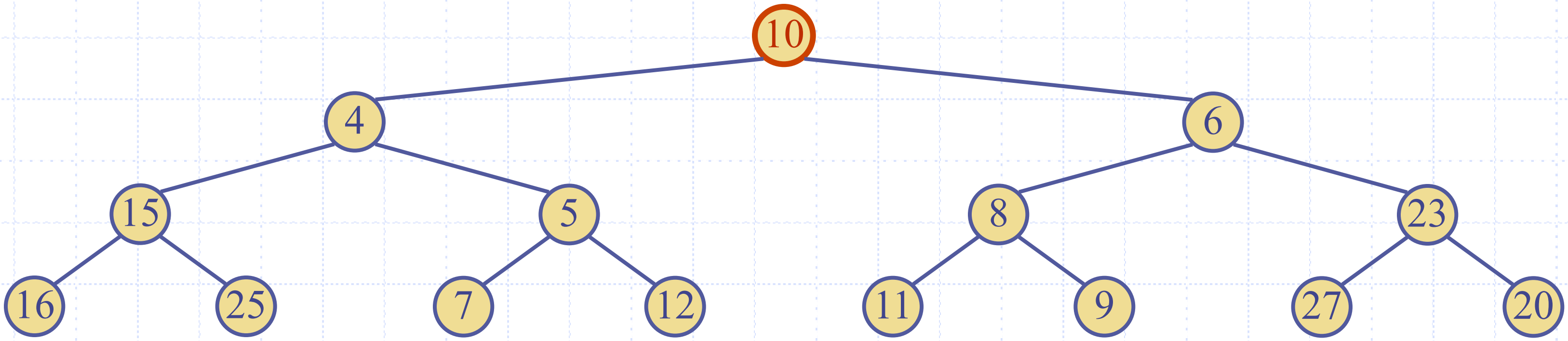


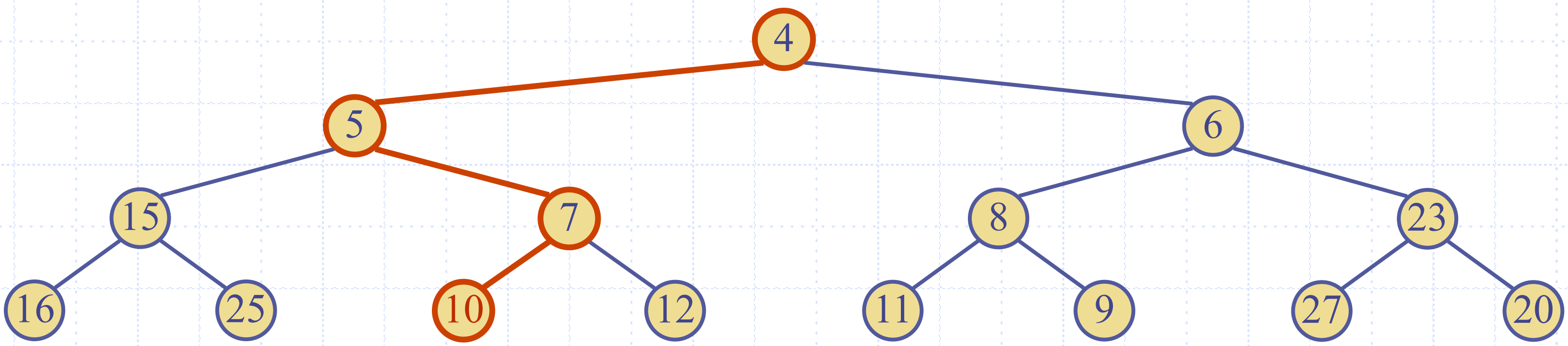


Example (contd.)



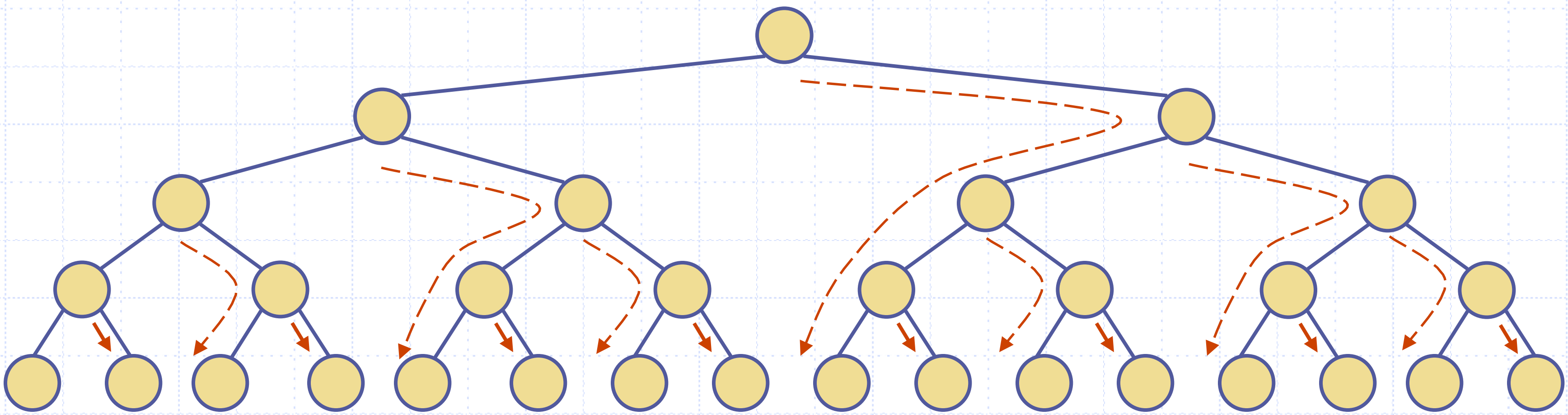
Example (end)





Analysis

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



In-class Exercise

What does the heap look like after the following sequence of insertions:

5 30 2 15 7 45 20 6 18

D-Heaps

Just like binary trees are not always optimal, binary heaps are not

D-Heaps are heaps that have D children at each node

Speeds insert/delete

Does mean more computation per node ..

Binomial Queues

Text

- Read Weiss, §6.8

Binomial Queue

- Definition of binomial queue
- Definition of binary addition

Building a Binomial Queue

- Sequence of inserts
- What in the world does binary addition have in common with binomial queues?

Motivation

A binary heap provides O(log n) inserts and O(log n) deletes but suffers from O(log n) merges

A binomial queue offers O(log n) inserts and O(log n) deletes *and* O(log n) merges

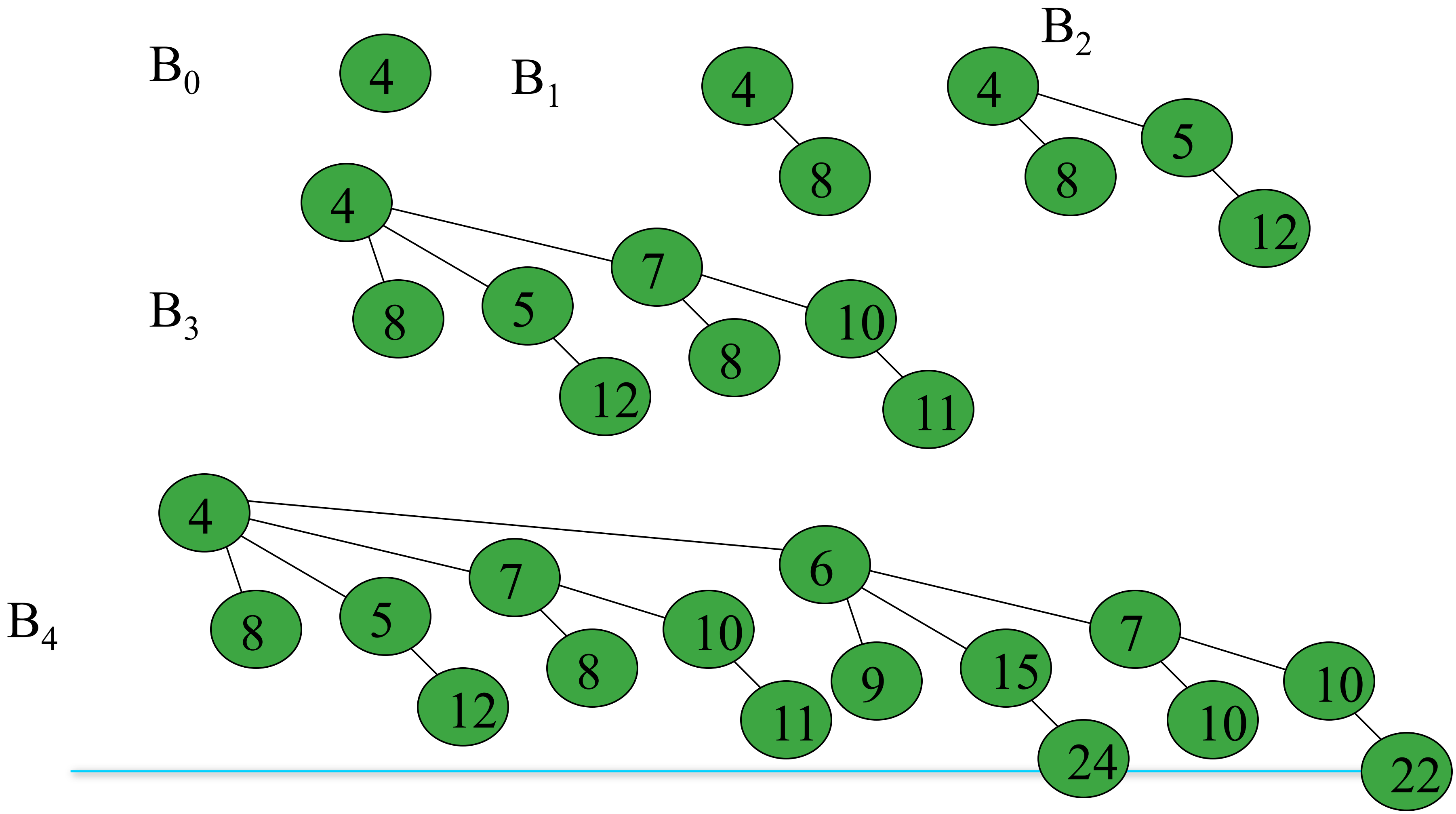
Note, however, binomial queue inserts and deletes are *more expensive* than binary heap inserts and deletes worst case (but constant time in average case)

Definition

A Binomial Queue is a collection of heap-ordered trees known as a forest. Each tree is a binomial tree. A recursive definition is:

1. A binomial tree of height 0 is a one-node tree.
2. A binomial tree, B_k, of height k is formed by attaching a binomial tree B_{k-1} to the root of another binomial tree B_{k-1} .

Examples



B_0
4
B_1
4
8
B_2
4
5
8
12
B_3
4
7
8
5
12
8
10
11
B_4
4
6
7
8
5
12
8
10
11
9
15
24
7
10
10
22

Questions

1. How many nodes does the binomial tree $\boldsymbol{B_k}$ have?

2. How many children does the root of $\boldsymbol{B_k}$ have?

3. What types of binomial trees are the children of the root of $\boldsymbol{B_k}$?

4. Is there a binomial queue with one node? With two nodes? With three nodes? … With n nodes for any positive integer n?

Binary Numbers

Consider binary numbers

Positional notation:

- 2^5 2^4 2^3 2^2 2^1 2^0
- 0 1 0 1 1 0
- 0 + 16 + 0 + 4 + 2 + 0 = 22

➔ $010110_2 = 22_{10}$

What is the decimal value of these binary numbers?

- 011 =
- 101 =
- 10110 =
- 1001011 =

Binary Numbers

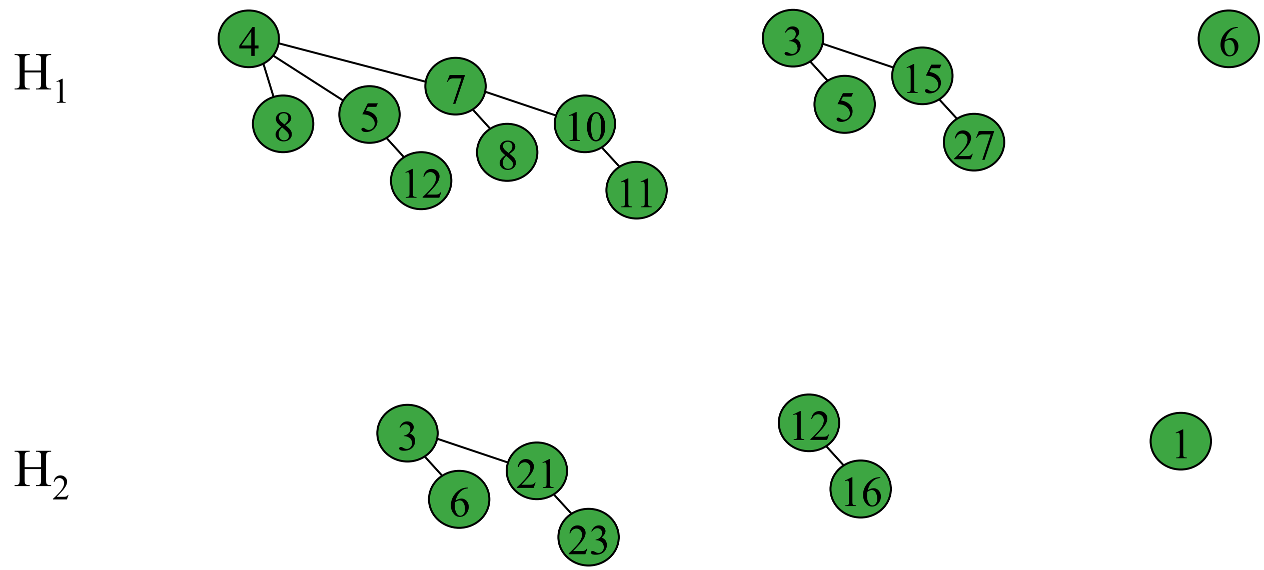
- Consider binary numbers
- Positional notation:
  - 2^5 2^4 2^3 2^2 2^1 2^0
  - 0 1 0 1 1 0
  - 0 + 16 + 0 + 4 + 2 + 0 = 22
  - ➔ $010110_2 = 22_{10}$
- What is the decimal value of these binary numbers?
  - 011 = 3
  - 101 = 5
  - 10110 = 22
  - 1001011 = 75

Binary Addition

```
(carry)     1   1

          1   0   1   1   0   1   0

      +   0   0   1   1   1   0   0

    ----------------------------------

          1   1   1   0   1   1   0
```

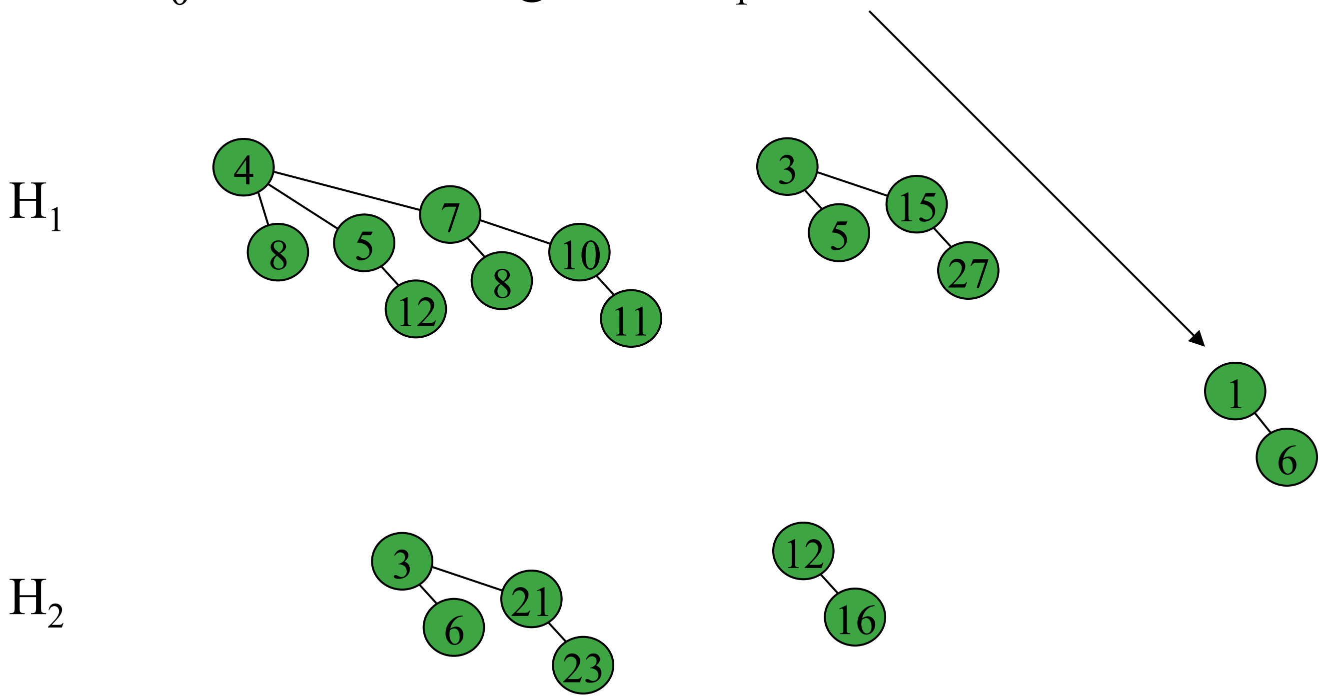
Merging Binomial Queues

Consider two binomial queues, H_1 and H_2



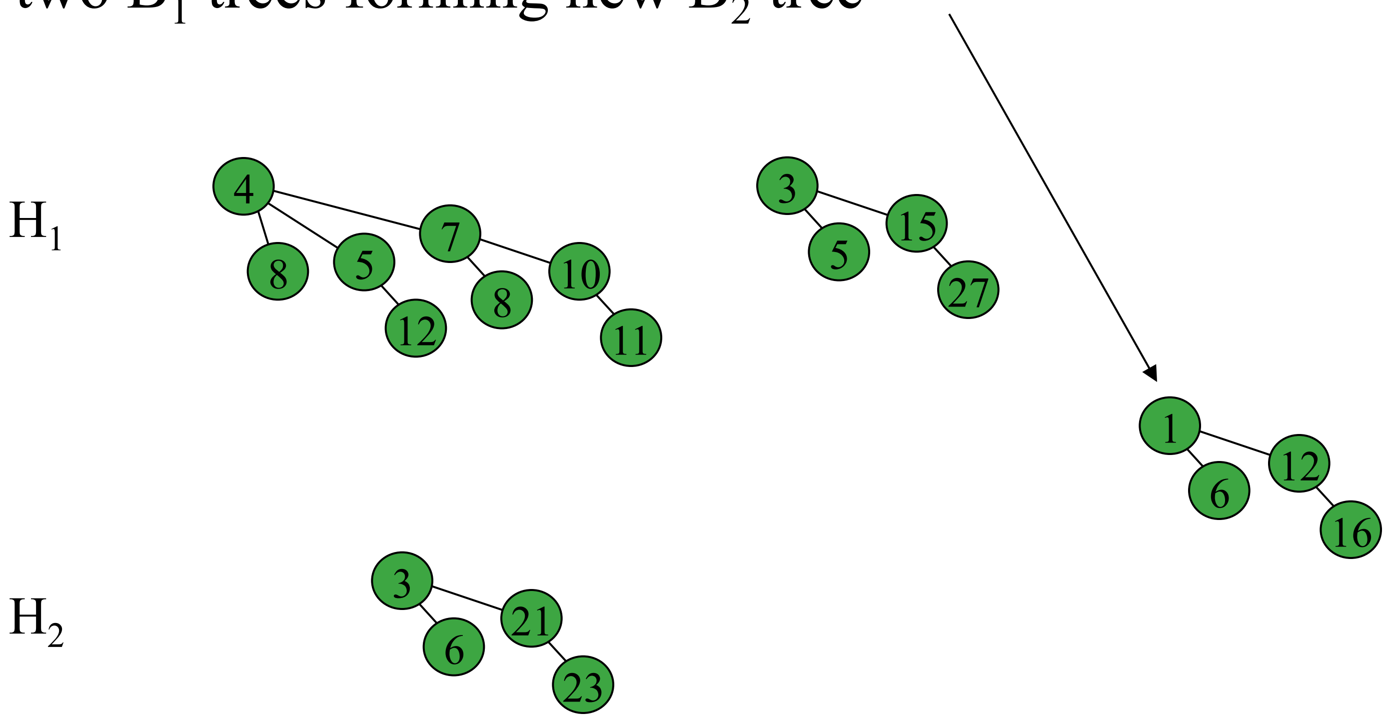
Merging Binomial Queues

Merge two B_0 trees forming new B_1 tree



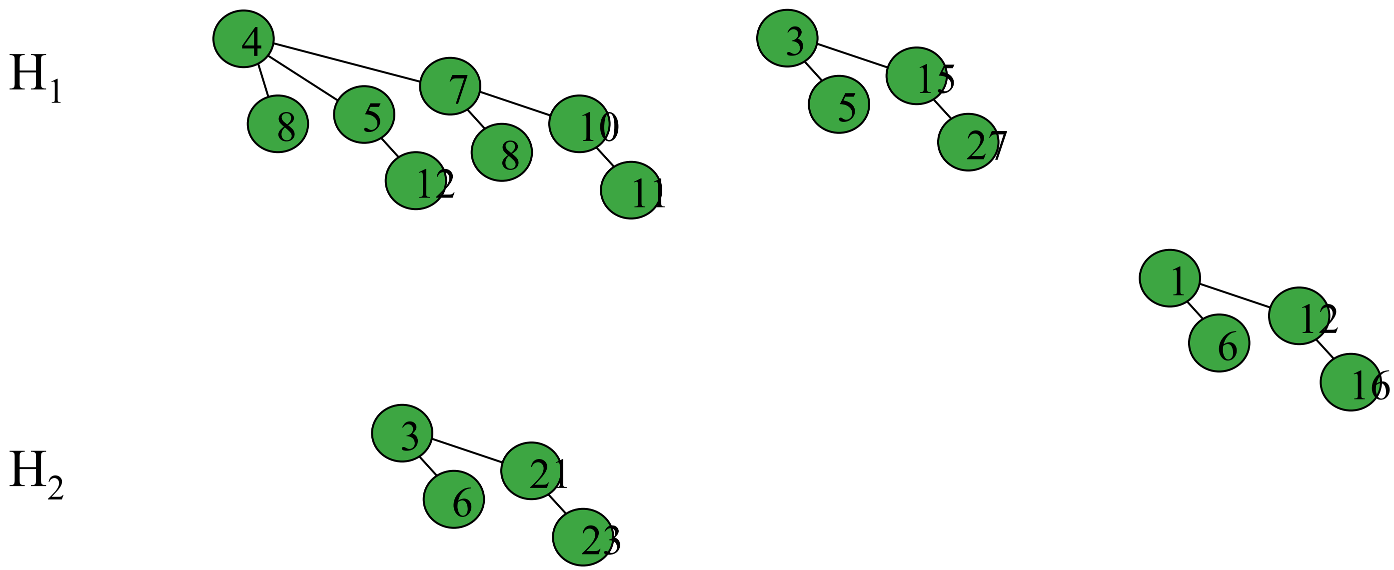
Merging Binomial Queues

Merge two B_1 trees forming new B_2 tree



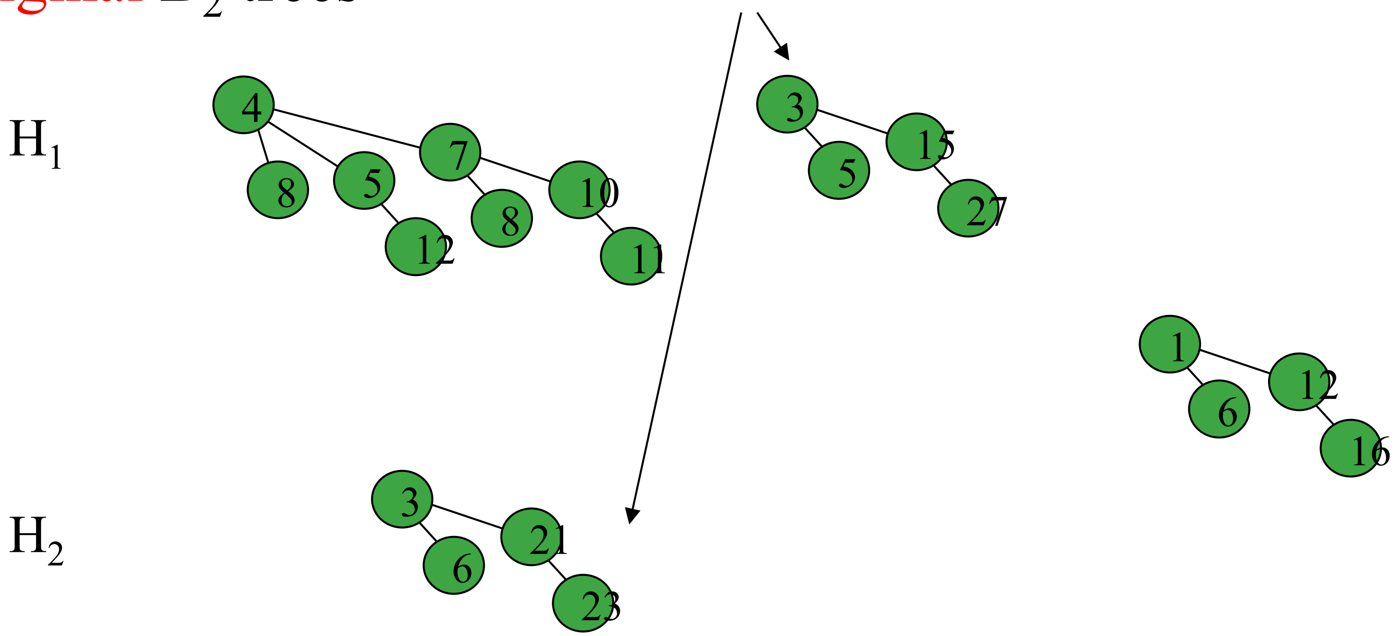
Merging Binomial Queues

Merge two B_2 trees forming new B_3 tree (but which two B_2 trees?)



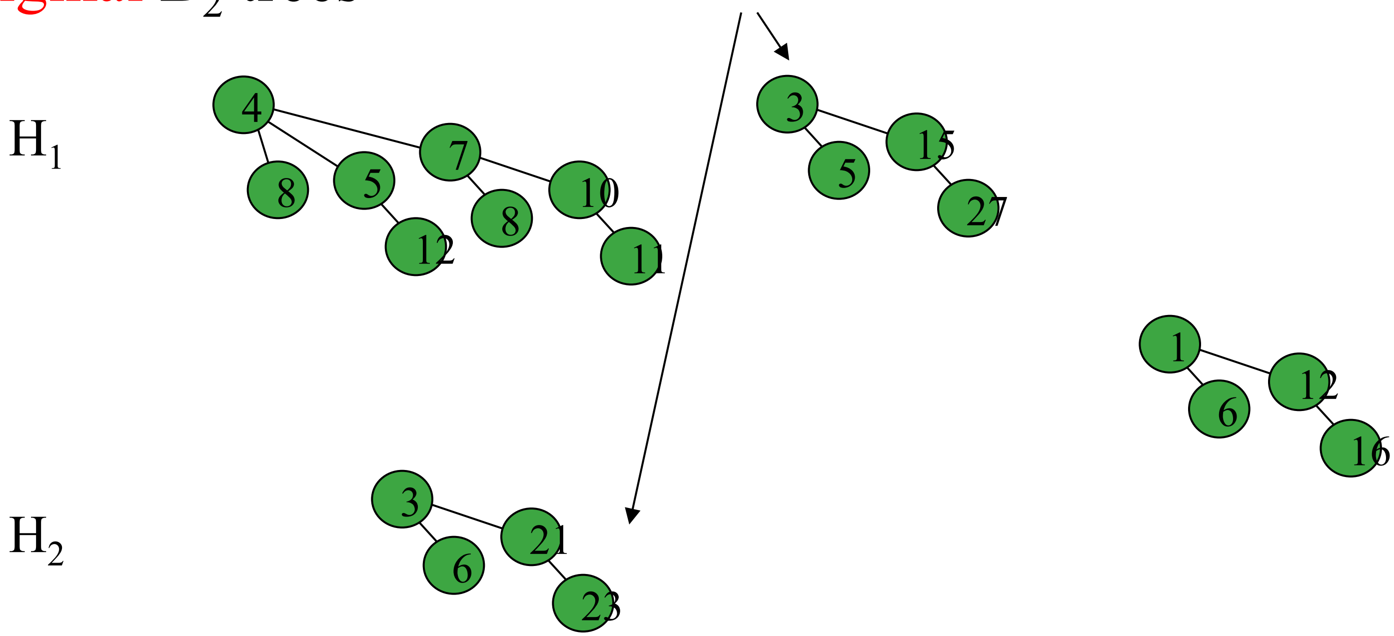
Merging Binomial Queues

Which two B_2 trees? Arbitrary decision: merge two original B_2 trees



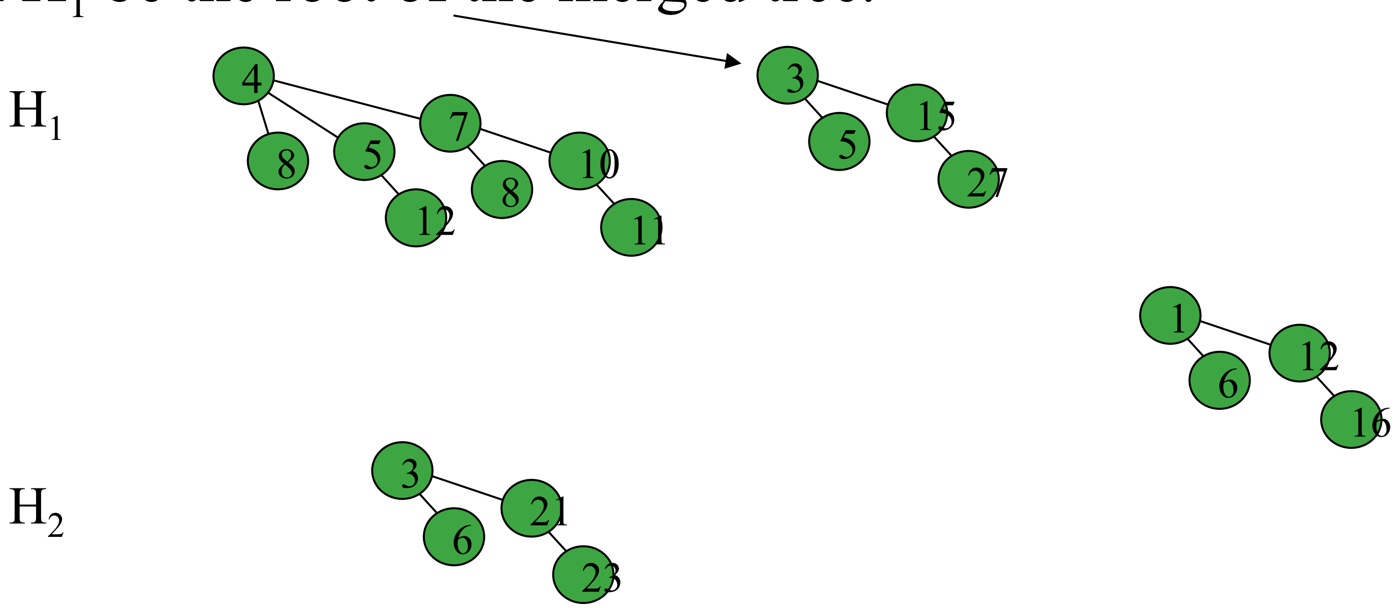
Merging Binomial Queues

Which two B_2 trees? Arbitrary decision: merge two original B_2 trees



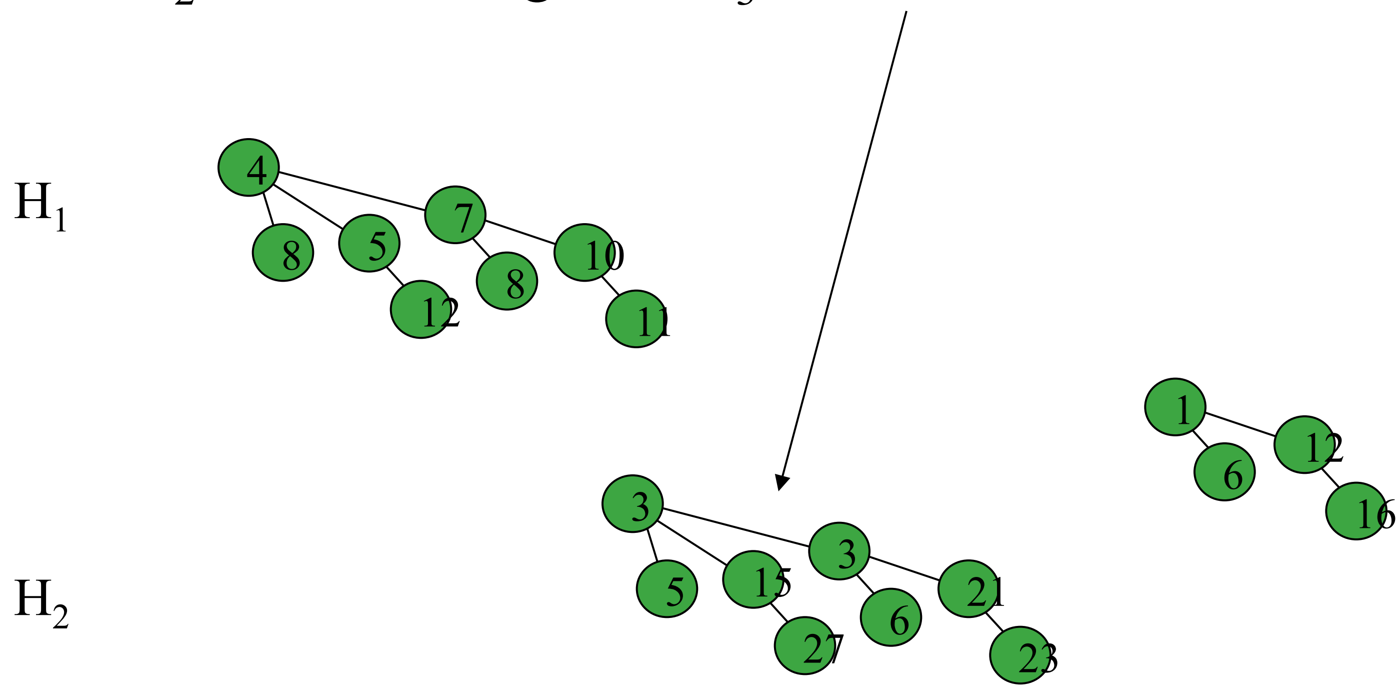
Merging Binomial Queues

Which root becomes root of merged tree?
Arbitrary decision: in case of a tie, make the root of H_1 be the root of the merged tree.



Merging Binomial Queues

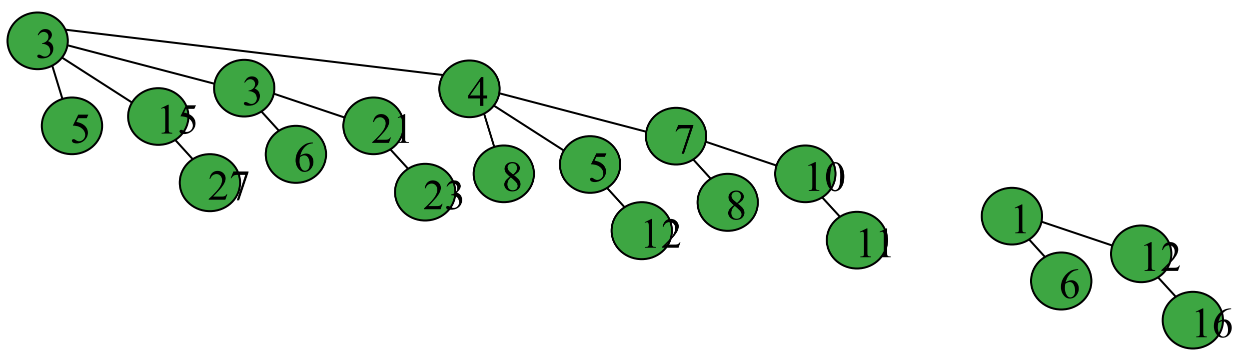
Merge two B_2 trees forming new B_3 tree



Merging Binomial Queues

Merge two B_3 trees forming a new B_4 tree

H_1

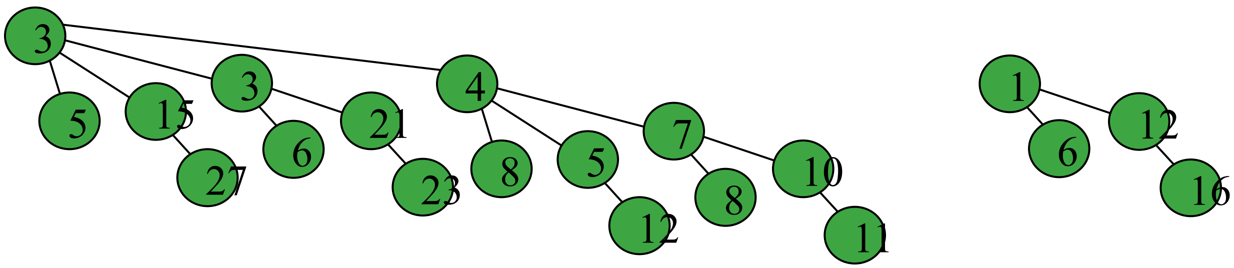


H_2

Merging Binomial Queues

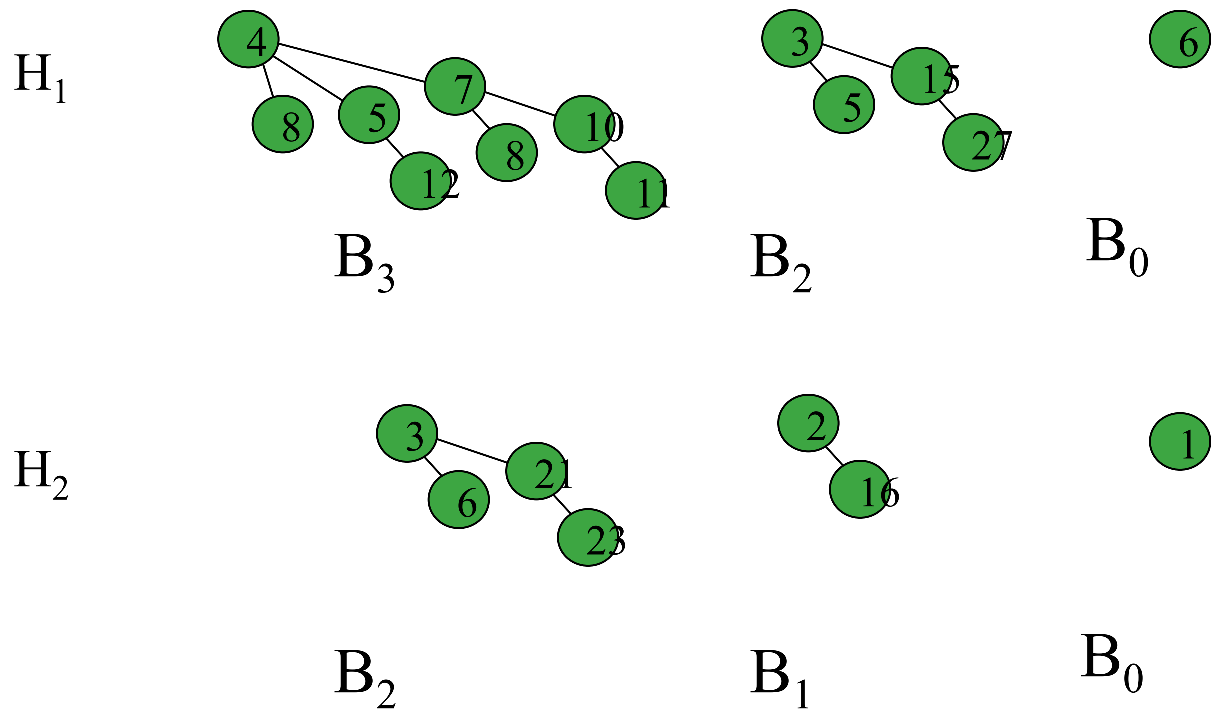
Call new binomial queue H_3

H_3



Merging Binomial Queues

Reconsider the two original binomial queues, H_1 and H_2 and identify types of trees

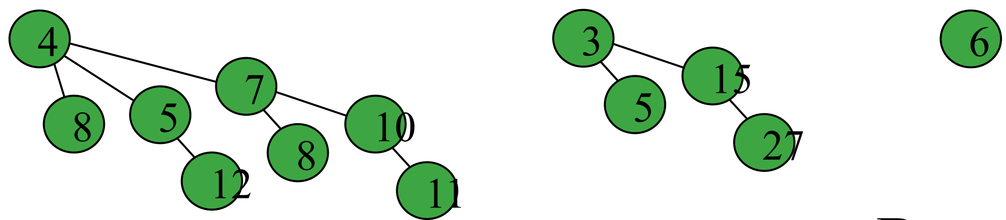


Merging Binomial Queues

Represent each binomial queue by a binary number

H_1

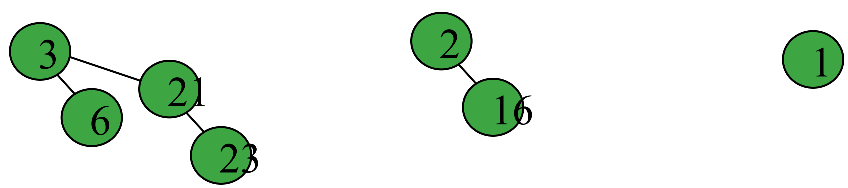
B_4	B_3	B_2	B_1	B_0
0	1	1	0	1



B_3 B_2 B_0

H_2

B_4	B_3	B_2	B_1	B_0
0	0	1	1	1



B_2 B_1 B_0

Merging Binomial Queues

Note that the merged binomial queue can be represented by the binary sum

H_1

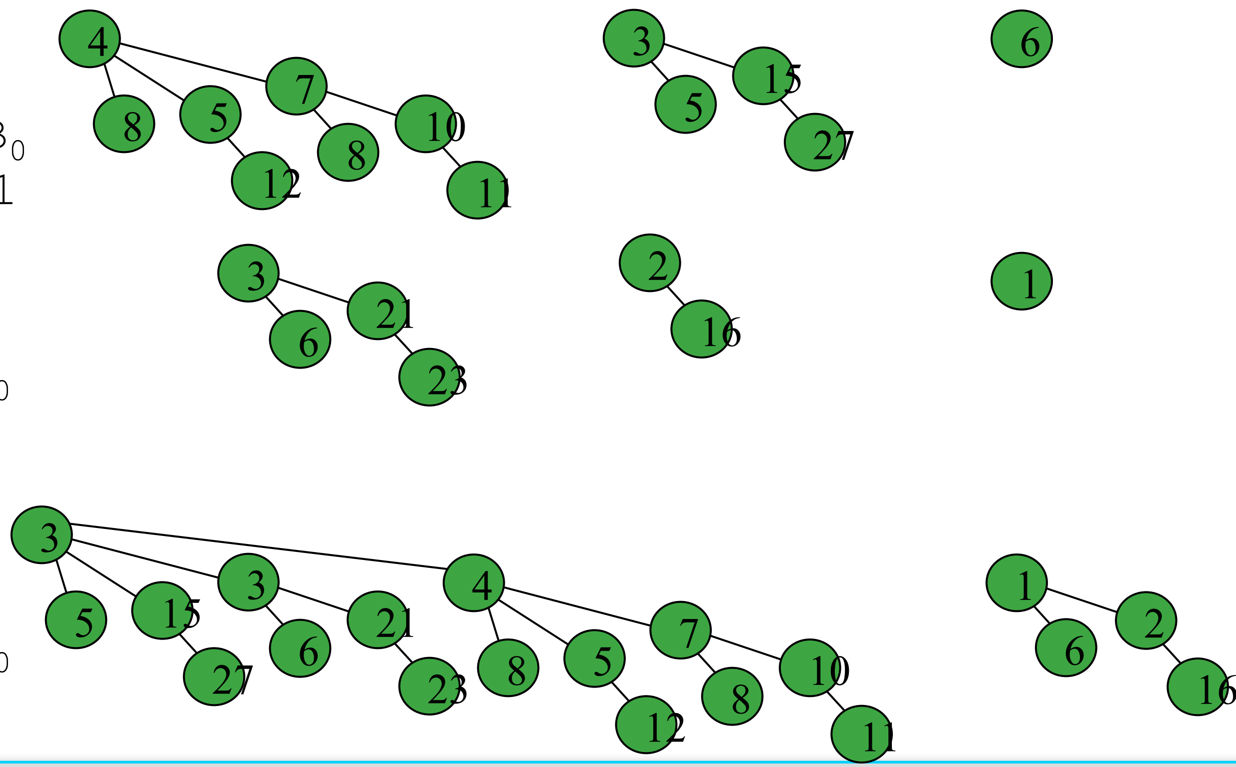
B_4	B_3	B_2	B_1	B_0
0	1	1	0	1

H_2

B_4	B_3	B_2	B_1	B_0
0	0	1	1	1

H_3

B_4	B_3	B_2	B_1	B_0
1	0	1	0	0



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Implementing Binomial Queues

This suggests a way to implement binomial queues:

1. Use a *k-ary* tree to represent each binomial tree – sibling and child pointers

2. Use a Vector to hold references to the root node of each binomial tree

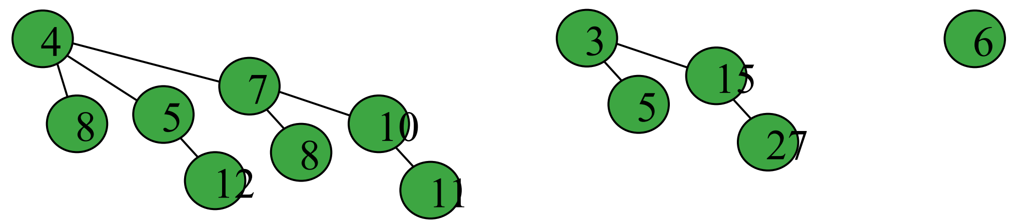
3. Keep a reference to smallest root for past find min (e.g. a Heap on positions).

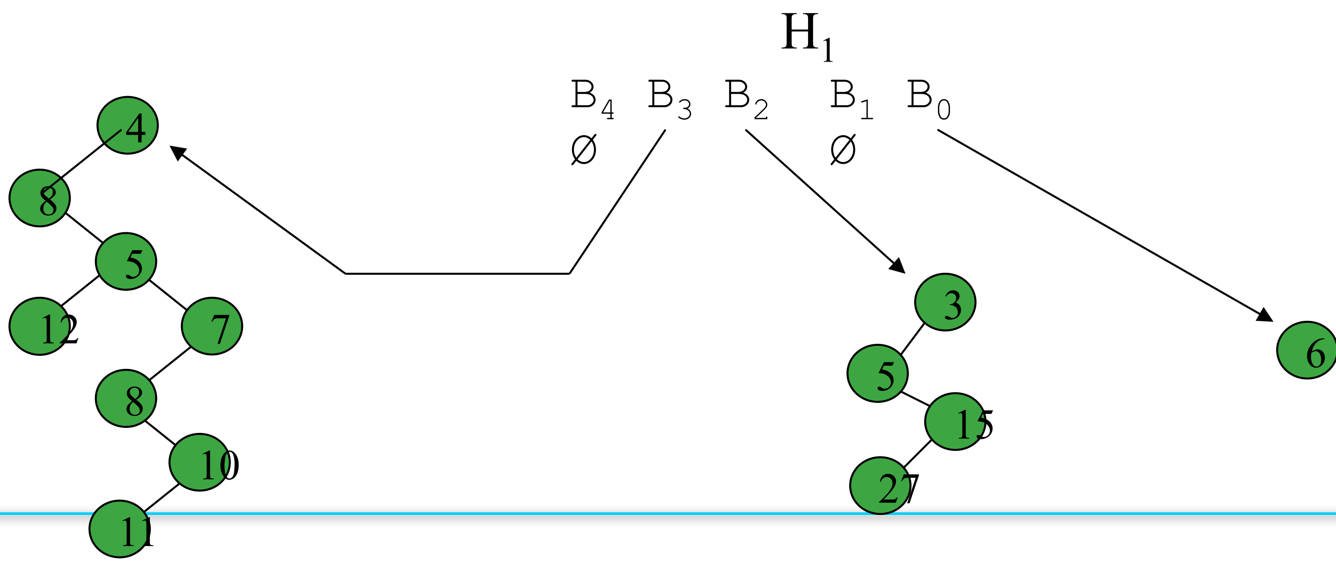
Implementing Binomial Queues

Use a k-ary tree to represent each binomial tree. Use an array to hold references to root nodes of each binomial tree.

H_1

B_4	B_3	B_2	B_1	B_0
0	1	1	0	1





Questions

We now know how to merge two binomial queues. How do you perform an *insert*?

How do you perform a *delete*?

What is the order of complexity of a *merge*? an *insert*? a *delete*?

Implementing Binomial Queues

Carefully study Java code in

Weiss, Figure 6.52 – 6.56

PQ summary

What is a PQ (what ops)

What are heap properties

How do heaps work

Binomial queues as a second example