1. For the Markov Chain given by:

\[ Q = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{pmatrix}, \]

(a) Prove that it is Ergodic.
(b) Compute its stationary distribution.
(c) Prove that it is reversible.
(d) Convert it into a reversible Markov Chain with stationary distribution \((\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{3}{8})\).

(a) We consider the random-walk graph corresponding to the given Markov Chain Q.
Since there exists a cycle \(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1\) which touches all vertices, Q is irreducible.
In addition, since every vertex is part of a cycle of length 2 and a cycle of length 3, the GCD of cycles through each vertex is 1, and the chain is therefore aperiodic.
As a result, Q is Ergodic.
(b) Let \(\Pi = (p_1, p_2, p_3, p_4)\) be the stationary distribution for Q. Then \(\Pi = \Pi Q\). This gives us four linear equations in \(p_1, p_2, p_3, p_4\). Solving these equations, along with \(p_1 + p_2 + p_3 + p_4 = 1\), we get \(\Pi = (\frac{1}{5}, \frac{3}{10}, \frac{1}{5}, \frac{3}{10})\).
(c) To show that Q is reversible, we must demonstrate that the detailed balance condition holds. If we compute \(\Pi_i Q_{ij}\) and \(\Pi_j Q_{ji}\) for \(i, j \in \{1, 2, 3, 4\}\), we get:

\[ \Pi_i Q_{ij} = \Pi_j Q_{ji} = \begin{cases} 
0 & i = j \vee i + j = 4 \\
\frac{1}{10} & \text{otherwise}
\end{cases} \quad (1) \]
Regardless, the two quantities are always equal, and the detailed balance condition is satisfied.

(d) As Q is not-symmetric, but reversible, we can apply the Metropolis-Hastings algorithm to transform the chain into one with the desired stationary distribution. We compute the new transition matrix according to the rule:

$$Q'_{ij} = Q_{ij} \cdot \min \left( 1, \frac{\Pi'_j Q_{ji}}{\Pi'_i Q_{ij}} \right)$$  \hspace{1cm} (2)

when \(i \neq j\), and have the diagonal elements comprise the remainder of the distribution. Applying this rule, we get:

$$Q' = \begin{pmatrix} 3/4 & 1/6 & 0 & 1/12 \\ 1/3 & 1/4 & 1/4 & 1/6 \\ 0 & 1/2 & 1/6 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}$$  \hspace{1cm} (3)

2. \(\tau_{\text{mix}}(\epsilon)\) is defined as the maximization of \(\tau_{\text{mix}}(q_0, \epsilon)\) over all initial distributions \(q_0\). Prove that this maximization can be limited to point distributions.

Since we can represent any distribution as a linear combination of point distributions, and since the \(L_1\) norm of two distributions is a convex distance function, it satisfies:

$$\|q, \Pi\|_1 = \left\| \sum_{i=1}^{n} c_i q_i, \Pi \right\|_1$$ \hspace{1cm} (4)

$$\leq \sum_{i=1}^{n} c_i \|q_i, \Pi\|_1$$ \hspace{1cm} (5)

$$\leq \max_i \|q_i, \Pi\|_1$$ \hspace{1cm} (6)

The last equation corresponds to \(q\) being a point distribution, as all but one of the \(c_i\) are 0. Therefore, point distributions yield the longest mixing times, and it is sufficient to examine only them.