1. For the Hamming Center Problem, after solving the LP relaxation, if we deterministically round each $x_i$ to 0 if $x_i < 0.5$ and to 1 if $x_i \geq 0.5$, prove that the resulting Hamming Radius is no more than twice the optimum Hamming Radius.

Let $X$ be the deterministically rounded form of $x$, the optimum solution to the relaxed LP. If $\text{dist}$ is the Hamming Distance function, let us define the following:

$$d_i = \text{dist}(A_i, x) = \sum_{A_{ij}=1} 1 - x_j + \sum_{A_{ij}=0} x_j$$

$$D_i = \text{dist}(A_i, X) = \sum_{A_{ij}=1} 1 - X_j + \sum_{A_{ij}=0} X_j$$

For any particular $i$,

$$D_i - d_i = \sum_{A_{ij}=1} (x_j - X_j) + \sum_{A_{ij}=0} (X_j - x_j)$$

In the first summation, we have:

$$0 \leq x_j < 0.5 \implies X_j = 0 \implies x_j - X_j = x_j < 1 - x_j \leq 1 - x_j$$

$$0.5 \leq x_j \leq 1 \implies X_j = 1 \implies x_j - X_j \leq 0 \leq 1 - x_j$$

In the second summation, we have:

$$0 \leq x_j < 0.5 \implies X_j = 0 \implies X_j - x_j = -x_j < 0 \leq x_j$$

$$0.5 \leq x_j \leq 1 \implies X_j = 1 \implies X_j - x_j = 1 - x_j \leq x_j$$
Combining these results, we have:

\[ D_i - d_i = \sum_{A_{ij} = 1} (x_j - X_j) + \sum_{A_{ij} = 0} (X_j - x_j) \]  
\[ \leq \sum_{A_{ij} = 1} 1 - x_j + \sum_{A_{ij} = 0} x_j \]
\[ = d_i \]
\[ D_i \leq 2d_i \]  

2. In the Lattice Approximation Problem, assume that for all \( i \), \( c_i \leq \frac{1}{4} \) or \( c_i \geq \frac{3}{4} \). How well would a deterministic rounding algorithm perform in this case?

Let us define the deterministically rounded vector \( X \) as:

\[ X_i = \begin{cases} 
0 & \text{if } c_i \leq \frac{1}{4}, \\
1 & \text{if } c_i \geq \frac{3}{4}.
\end{cases} \]  

Let \( Z = A(X - C) \). For any \( i \),

\[ |Z_i| = \left| \sum_{j=1}^{n} A_{ij}(X_j - c_j) \right| \]
\[ = \sum_{1 \leq j \leq n, c_j \leq \frac{1}{4}} |A_{ij}(0 - c_j)| + \sum_{1 \leq j \leq n, c_j \geq \frac{3}{4}} |A_{ij}(1 - c_j)| \]
\[ = \sum_{1 \leq j \leq n, c_j \leq \frac{1}{4}} A_{ij}c_j + \sum_{1 \leq j \leq n, c_j \geq \frac{3}{4}} A_{ij}(1 - c_j) \]
\[ \leq \sum_{1 \leq j \leq n, c_j \leq \frac{1}{4}} A_{ij} \cdot \frac{1}{4} + \sum_{1 \leq j \leq n, c_j \geq \frac{3}{4}} A_{ij} \cdot \frac{1}{4} \]
\[ \leq \frac{1}{4} \cdot \sum_{1 \leq j \leq n} A_{ij} \]
\[ \leq \frac{n}{4} \]  

The deterministic rounding algorithm is at most \( n/4 \) worse than the optimal solution.

3. We can define the concept of the “best” fraction in a few different ways:

- The fraction that gives us a tighter bound on the probability that quicksort terminates within \( 16n \ln n \) comparisons.
• The fraction that gives us a tighter bound on the number of comparisons that we can prove that quicksort terminates within, with high probability.

Arguably, the latter bound is the more useful one.

If $X_{ij}$ is the random variable indicating that the pivot on the path for element $i$, at level $j$, fell in the middle $r^{th}$ fraction, then the children of that level receive at most $(1 + r)/2$ of the parent’s input size.

If $k_i$ is the number of times the pivot falls in this fraction for a given element $i$, then following the standard proof,

$$
\left( \frac{1 + r}{2} \right)^{k_i} > 1
$$

$$
k_i < \frac{\ln n}{\ln \frac{2}{r+1}}
$$

$$
def = q \ln n
$$

If $X = X_{i1} + \cdots + X_{ic \ln n}$, then $E[X_i] = rc \ln n$, where $c$ is the maximum depth we are willing to tolerate.

Using the form of the Chernoff-Hoeffding bound given in Theorem 31 (c) (ii), we get:

$$
P[X_i \leq q \ln n] \leq \exp \left( - \frac{(rc \ln n - q \ln n)^2}{2c \ln n} \right)
$$

To bound this probability, we must prove:

$$
\exp \left( - \frac{(rc \ln n - q \ln n)^2}{2c \ln n} \right) \leq \frac{1}{n^b}
$$

Simplifying,

$$
\exp \left( - \frac{(rc \ln n - q \ln n)^2}{2c \ln n} \right) \leq \frac{1}{n^b}
$$

$$
(rc - q)^2 \geq 2bc
$$

We have a single equation with unknowns $r$, $c$ and $b$. Given two of these, we can fix the third.

For $r = 3/4$, we can bound the number of comparisons we can prove happening with probability at most $1/n^2$ ($b = 2$) by solving for $c$ in the above equation, which gives us $c \geq 22.7$. Depending on which form of the Chernoff Bound was used, other values might be obtained. In any case, this bound is worse than that obtained when $r = 1/2$. 

3
4. We can generalize the problem in a number of ways. One such is to use the following observations:

- The values of $c_i$ are known to us, so we can use them in our rounding procedure.
- For the best bound, the expected rounding should still be $c_i$, for each $i$.

In the general case, if we divide the interval $[0, 1]$ into $k$ divisions, then we round to some fraction $j/k$ depending on the division within which $c_i$ falls. If $j/k \leq c_i < (j + 1)/k$, then

\[
\begin{align*}
P[X_i = j/k] &= 1 + j - kc_i \\
P[X_i = (j + 1)/k] &= kc_i - j \\
E[X] &= \frac{j}{k} \cdot (1 + j - kc_i) + \frac{j + 1}{k} \cdot (kc_i - j) \\
&= \frac{kc_i}{k} \\
&= c_i
\end{align*}
\]

If we are allowed to round to one of $\{0, 1/2, 1\}$, we have just the two cases:
If $0 \leq c_i < 1/2$,

\[
\begin{align*}
P[X_i = 0] &= 1 - 2c_i \\
P[X_i = 1/2] &= 2c_i
\end{align*}
\]

If $1/2 \leq c_i \leq 1$,

\[
\begin{align*}
P[X_i = 1/2] &= 2 - 2c_i \\
P[X_i = 1] &= 2c_i - 1
\end{align*}
\]

The Chernoff Bound as it stands will not give us any better a bound with this type of rounding, since it only makes use of the expectation of the rounded value, not a measure of its skew. However, other methods can be used to show that the algorithm performs better as $k$ increases.