A useful formula

- Chernoff bounds for 0,1 valued r.v.s:

\[
\Pr[X \geq (1 + \delta)\mu] \leq \begin{cases} 
  e^{-\frac{1}{3}\mu\delta^2} & \text{if } \delta \leq 1 \\
  e^{-\frac{1}{8}\mu\delta\ln \delta} & \text{if } \delta > 1 
\end{cases}
\]

I. In an $n \times n$ array $A$, a monotone connected path of length $k$ consists of locations

$(i_1, j_1), (i_2, j_2), \cdots, (i_{k+1}, j_{k+1})$ such that for $1 \leq \ell \leq k$, $(i_{\ell+1}, j_{\ell+1}) \in \{(i_{\ell} + 1, j_{\ell}), (i_{\ell}, j_{\ell} + 1)\}$.

Now throw $n^2$ balls into the array locations independently and uniformly at random. Prove that there exists a constant $c > 0$ such that with high probability every connected path of length $\ln n$ has no more than $c \ln n$ balls.

II. Let $n$ be a power of 2, and let $n = 2^k$. Let the r.v.s $X_1, X_2, \cdots, X_k$ be independent, 0, 1-valued, and uniformly distributed. For any $A \subseteq \{1, 2, \cdots, k\}$ and $|A| \geq 1$, define the r.v. $Y_A = \bigoplus_{i \in A} X_i$.

Prove that the $n - 1$ $Y$ r.v.s are pairwise independent and uniformly distributed.

III. Consider the following generalized form of the set discrepancy problem. Given an $n \times n$ array $A$ in which each array element is a non-negative integer of value no more than $n$, as in the standard problem, $X_i$s are chosen independently and uniformly at random from $\{-1, +1\}$. It has already been established that

\[
\sum_{i=1}^n P[A_i X \geq c] + \sum_{i=1}^n P[A_i X \leq -c] < 1,
\]

for some value $c$. Describe a derandomization of this algorithm by the conditional probability method.