600.363/463 Key to Mid Sem 2

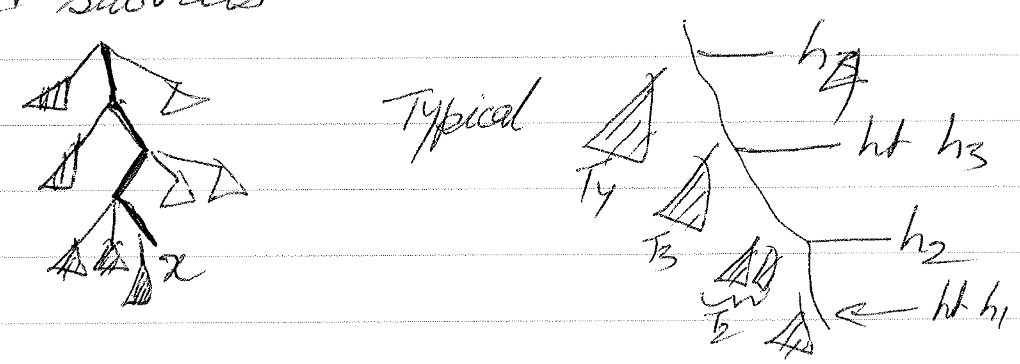
I

1. Yes. In n^3+10n^2, the dominant term is n^3.
   In $\frac{1}{100}n^3+100n$, the dominant term is n^3.
   Hence the equality holds.

2. Yes. $n^3\log n$ grows faster than $n^{2.5}\log^5 n$.
   ~~clearly~~ $n^{0.5}$ beats any $\log^k n$.

3. Yes. 3^n grows much faster than $n^3 2^n$.
   $3^n = 2^n(1.5)^n$; $\underbrace{1.5^n}_{\text{exponential}}$ grows faster than $\underbrace{n^3}_{\text{polynomial}}$.

II As a typical example, consider ~~Red~~ 2,3,4 trees.
Locate x, as shown below & we need combine the hatched subtrees.



First combine ~~with~~ T_1 with T_2 in $c(h_2-h_1)$ steps.
Results in depth h_2 or h_2+1.

Combine the resulting tree with T_3 in $c(h_3-h_2)$ steps.
Results in depth h_3 or h_3+1.

then combine with T_4 in $c(h_4-h_3)$ steps.

Total # of steps $= c(h_2-h_1)+c(h_3-h_2)+c(h_4-h_3)$
$\leq ch_4$
$= O(\log n)$

III Yes.

Proof: $wt(e_1) > wt(e_2) > \dots > wt(e_m)$

By induction, we prove that the edges retained must be in the MST, and the edges deleted cannot be in the MST.

Base: When e_1 is considered, if deletion of e_1 disconnects G, then e_1 is in every spanning tree & hence it has to be in the MST of G.

If deletion of e_1 doesn't disconnect G, let T be any MST of G which includes e_1. Then include any edge in G which forms a cycle that includes e_1 (otherwise deletion of e_1 disconnects G) & remove e_1. The new tree is of weight less than the wt of T - hence T cannot ~~be~~ be an MST.

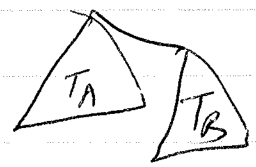
Inductive step

$$\overset{\text{current}}{\downarrow}$$
$$\dots \underset{t_1}{\underline{\ \ }}\ \underset{t_2}{\underline{\ \ }} \dots \underset{t_l}{\underline{\ \ }} \dots e_k\ e_{k+1} \dots e_m$$

At any stage, when e_k is being considered, assume that edges $t_1, t_2, \dots, t_l$ have ~~are~~ already been chosen ~~to~~ to be in the MST. If e_k disconnects the graph, then e_k has to be in the MST (as above). If e_k doesn't disconnected and if e_k is in the MST, argue that inclusion of another edge from $e_{k+1}, \dots, e_m$ forms a cycle ~~with~~ that includes e_k (and some t_i's). Then include that edge & remove e_k ~~too~~ --- resulting in a smaller wt spanning tree.

IV At any time, let $f(h)$ be the min size of a tree of depth h.

When UNION(A,B,C) results in T_C of height h & is of min size, then

$T_C =$ 

$size(T_A) = \frac{1}{2} f(h-1)$ and $size(T_B) = f(h-1)$.

Hence $f(h) = f(h-1) + \frac{1}{2} f(h-1) = \frac{3}{2} f(h-1)$

Also $f(0) = 1$.

Then $f(h) = \left(\frac{3}{2}\right)^h$.

If we define $\hat{F}$ & $\hat{G}$ as:

$\hat{F}(h) = 1.5^{\hat{F}(h-1)}$

& $\hat{G}$ is the inverse of $\hat{F}$.

We can show that $\hat{G}(n) = O(G(n)) = O(\log^* n)$.

Replace F by $\hat{F}$ in the standard proof, resulting in $O(n \log^* n)$ speed.