Lecture 9. Machine Learning: Structure & Latent.

Note Title 2/4/2010

Structure Max-Margin extends binary-classification methods so they can be applied to learn the parameters of an MRF, HMM, SCFG or other model.

Recall standard SVM. for binary classification.

$$R(\underline{\lambda}) = \frac{1}{2}|\underline{\lambda}|^2 + C\sum_{i=1}^{m} \max\{0, 1 - y_i \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i)\}$$

Training Data $\{(y_i, \underline{d}_i)\}$ $\quad y_i \in \{\pm 1\}$

e.g. to get a plane, set $\underline{\phi}(\underline{d}) = \underline{d}$.

Decision rule: $\hat{y}_i(\underline{\lambda}) = \arg\max_y y\,\underline{\lambda}\cdot\underline{\phi}(d_i) = \operatorname{sgn} \underline{\lambda}\cdot\underline{\phi}(d_i)$.

The task is to minimize $R(\underline{\lambda})$ w.r.t. $\underline{\lambda}$ which maximizes the 'margin' $1/|\underline{\lambda}|$.

Here is a more general formulation that can be used if the output variable y is a vector $\underline{y} = (y_1 \cdots y_N)$ — i.e. it could be the state of an MRF, an HMM, or a SCFG.

$$R(\underline{\lambda}) = \frac{1}{2}|\underline{\lambda}|^2 + C\sum_{i=1}^{m} \Delta(\underline{y}_i; \hat{\underline{y}}_i(\underline{\lambda}))$$

decision rule: $\quad \hat{\underline{y}}_i(\underline{\lambda}) = \arg\max_{\underline{y}} \underline{\lambda}\cdot\underline{\Phi}(\underline{d}, \underline{y})$.

the error function $\Delta(\underline{y}_i; \hat{\underline{y}}_i(\underline{\lambda}))$ is any measure of distance between the true solution $\underline{y}_i$ and the estimate $\hat{\underline{y}}_i(\underline{\lambda})$.

→ to obtain binary-value. → (i) set $\underline{y}_i = y_i \in \{-1, 1\}$

(ii) $\underline{\Phi}(\underline{d}, y) = y\,\underline{\Phi}(\underline{d})$

(iii) $\Delta(y_i; \hat{y}_i(\lambda)) = \max\{0, 1 - y_i \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i)\}$

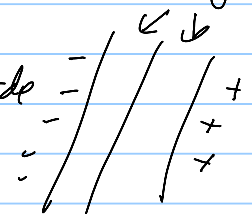
Hinge loss ——✓ because the function is 0

if $\quad y_i \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i) > 1$ (i.e. point is on the right side of the margin)

and the function increases linearly with $\underline{\lambda}\cdot\underline{\phi}(\underline{d}_i)$

(iv) $\quad \hat{y}_i(\underline{\lambda}) = \arg\max_y y\,\underline{\lambda}\cdot\underline{\phi}(\underline{d})$

margin



$\underline{\lambda}\cdot\underline{\phi}(\underline{d}) = 0$

(2) This more general formulation is $P(\underline{y},\underline{d}) = \frac{1}{Z} e^{\underline{\lambda}\cdot\underline{\phi}(\underline{y},\underline{d})}$

$$R(\underline{\lambda}) = \frac{1}{2}|\underline{\lambda}|^2 + C \sum_{i=1}^{m} \Delta(\underline{y}_i : \underline{\hat{y}}_i(\underline{\lambda}))$$

$$\underline{\hat{y}}_i = \arg\max_{\underline{y}} \underline{\lambda}\cdot\underline{\phi}(\underline{y},\underline{d})$$

(*) This requires an inference algorithm, like in the last few lectures.

for binary classification inference only has to compute $\max_{y_i \in \{\pm 1\}} y_i \underline{\lambda}\cdot\underline{\phi}(\underline{d})$ so is trivial.

(*) Also need to be able to minimize $R(\underline{\lambda})$ to find $\underline{\lambda}$ → hard because the error term $\Delta(\underline{y}_i : \underline{\hat{y}}_i(\underline{\lambda}))$ is a highly complicated function of $\underline{\lambda}$.

Modify $R(\underline{\lambda})$ to an upper bound $\overline{R}(\underline{\lambda})$

$$\overline{R}(\underline{\lambda}) = \frac{1}{2}|\underline{\lambda}|^2 + C \sum_{i=1}^{m} \{ \max_{\underline{y}} \{ \Delta(\underline{y}_i ; \underline{y}) + \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i ; \underline{y}) \} - \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i ; \underline{y}_i) \}$$

which is convex in $\underline{\lambda}$, hence has a single minimum.

To get this bound use two steps:

(step 1) $\max_{\hat{y}} \{ \Delta(\underline{y}_i : \underline{\hat{y}}) + \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i , \underline{\hat{y}}) \} \geq \Delta(\underline{y}_i ; \underline{\hat{y}}_i(\underline{\lambda})) + \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i , \underline{\hat{y}}_i(\underline{\lambda}))$

→ $\underline{\hat{y}}_i(\underline{\lambda})$ maximizes $\underline{\lambda}\cdot\underline{\phi}$

→ equality if it also maximizes $\Delta + \underline{\lambda}\cdot\underline{\phi}$

(step 2) $\underline{\lambda}\cdot\underline{\phi}(\underline{d}_i , \underline{\hat{y}}_i(\underline{\lambda})) \geq \underline{\lambda}\cdot\underline{\phi}(\underline{d}_i , \underline{y}_i)$

Note: bounds are 'tight' because if we can find a good solution then $\underline{y}_i \approx \underline{\hat{y}}_i(\underline{\lambda})$.

How to minimize $\overline{R}(\underline{\lambda})$?

Several Algorithms (hot topic)

Some in dual space — like original SVM for binary problem.

Simple: stochastic gradient descent

pick example $(\underline{d}_i , \underline{y}_i)$

take derivative of $\overline{R}(\underline{\lambda})$ w.r.t. $\underline{\lambda}$.

$$\underline{\lambda}^{t+1} = \underline{\lambda}^{t} - \epsilon C \{ \underline{\phi}(\underline{d}_i , \underline{\hat{y}}) - \underline{\phi}(\underline{d}_i , \underline{y}_i) \}$$

where $\underline{\hat{y}} = \text{ARGMAX} \{ \Delta(\underline{y}_i , \underline{y}) + \underline{\lambda}\cdot\underline{\phi}(\underline{x}_i , \underline{y}) \}$

Note: inference algorithm must be adapted to compute this.

(3) How to extend to models with latent (hidden) variables? Denote these variables by $\underline{h}$.

Want decision rule

$$(\hat{\underline{y}}, \hat{\underline{h}}) = \arg\max_{(\underline{y},\underline{h}) \in \mathcal{Y} \times H} \underline{\lambda} \cdot \underline{\phi}(\underline{d}, \underline{y}, \underline{h})$$ ← must be computable by inference algorithm

Training data $\{(\underline{d}^i, \underline{y}^i) : i = 1 \text{ to } N\}$. the hidden variables are not known.

loss function $\Delta(\underline{y}_i ; \hat{\underline{y}}_i(\underline{\lambda}), \hat{\underline{h}}_i(\underline{\lambda}))$

depends on the truth $\underline{y}_i$

the estimate of $\hat{\underline{y}}_i(\underline{\lambda}), \hat{\underline{h}}_i(\underline{\lambda})$ from the model

$$R(\underline{\lambda}) = \tfrac{1}{2}|\underline{\lambda}|^2 + C \sum_{i=1}^{m} \Delta(\underline{y}_i : \hat{\underline{y}}_i(\underline{\lambda}), \hat{\underline{h}}_i(\underline{\lambda}))$$

non-trivial function of $\underline{\lambda}$

replace $R(\underline{\lambda})$ by

$$\tilde{R}(\underline{\lambda}) = \tfrac{1}{2}|\underline{\lambda}|^2 + C \sum_{i=1}^{m} \left\{ \max_{(\hat{\underline{y}}, \hat{\underline{h}})} \{ \Delta(\underline{y}_i ; \hat{\underline{y}}, \hat{\underline{h}}) + \underline{\lambda} \cdot \underline{\phi}(\underline{d}_i ; \hat{\underline{y}}, \hat{\underline{h}}) \} - \max_{\underline{h}} \underline{\lambda} \cdot \underline{\phi}(\underline{d}_i, \underline{y}_i, \underline{h}) \right\}$$

$$\tilde{R}(\underline{\lambda}) = \underbrace{f(\underline{\lambda})}_{\text{convex}} + \underbrace{g(\underline{\lambda})}_{\text{concave}}, \quad \text{with } g(\underline{\lambda}) = -\max_{\underline{h}} \underline{\lambda} \cdot \underline{\phi}(\underline{d}_i, \underline{y}_i, \underline{h})$$

To show convexity and concavity

suppose $\tau(\underline{\lambda}) = \sum_{i=1}^{n} \max_{\hat{y}_i} \underline{\lambda} \cdot \underline{\Phi}(\underline{d}_i, \hat{\underline{y}}_i)$

convex if $\tau(\alpha \underline{\lambda}_1 + (1-\alpha)\underline{\lambda}_2) \leq \alpha \tau(\underline{\lambda}_1) + (1-\alpha)\tau(\underline{\lambda}_2)$

$$\tau(\alpha \underline{\lambda}_1 + (1-\alpha)\underline{\lambda}_2) = \sum_{i=1}^{n} \max_{\hat{y}_i} \{ (\alpha \underline{\lambda}_1 + (1-\alpha)\underline{\lambda}_2) \cdot \underline{\phi}(\underline{d}_i, \hat{\underline{y}}_i) \}$$

$$\alpha \tau(\underline{\lambda}_1) + (1-\alpha)\tau(\underline{\lambda}_2) = \alpha \sum_{i=1}^{n} \max_{y_i} \underline{\lambda}_1 \cdot \underline{\phi}(\underline{d}_i, \hat{\underline{y}}_i) + (1-\alpha) \sum_{i=1}^{n} \max_{y_i} \underline{\lambda}_2 \cdot \underline{\Phi}(\underline{d}_i, \hat{\underline{y}}_i)$$

but

$$\max_{\hat{y}_i} \alpha \underline{\lambda}_1 \cdot \underline{\phi}(\underline{x}_i, \hat{\underline{y}}_i) + \max_{\hat{y}_i} (1-\alpha)\underline{\lambda}_2 \cdot \underline{\phi}(\underline{x}_i, \hat{\underline{y}}_i) \geq \max_{\hat{y}_i} \{ (\alpha \underline{\lambda}_1 + (1-\alpha)\underline{\lambda}_2) \cdot \underline{\phi}(\underline{d}^i ; \hat{\underline{y}}_i) \}$$

hence $f(.)$ is convex

and $g(.)$ is concave.

(4) Apply CCCP algorithm.

Two stages: step 1. $\frac{\partial g(\underline{\lambda})^t}{\partial \underline{\lambda}} = -\phi(\underline{d}_i, \underline{y}_i, \underline{h}^*)$ where $\underline{h}^* = \arg\max_{\underline{h}} \underline{\lambda}^t \cdot \underline{\phi}(\underline{d}_i, \underline{y}_i, \underline{h})$

$\underline{\lambda}^t$ current estimate of $\underline{\lambda}$.

step 2. solve

$$\underline{\lambda}^{t+1} = \arg\min_{\lambda} \left\{ f(\underline{\lambda}) + \underline{\lambda} \cdot \frac{\partial}{\partial \underline{\lambda}} g(\underline{\lambda}^t) \right\}$$

This reduces to a modified SVM with known state

$$-\min_{\underline{\lambda}} \frac{1}{2} |\underline{\lambda}|^2 + C \sum_{i=1}^{n} \max_{(y,h)} \{ \underline{\lambda} \cdot \underline{\phi}(\underline{d}_i, y, h) + \Delta(y_i, y, h) \}$$
$$- C \sum_{i=1}^{n} \underline{\lambda} \cdot \underline{\phi}(\underline{d}_i, \underline{y}_i, \underline{h}_i^*)$$

Note: similarities to EM.

→ step 1 involves estimating the hidden state h_i^*

→ step 2 estimates $\underline{\lambda}$

repeat.

Note: like EM there is no guarantee that this will converge to the global optimum.