Statistics 202C. Spring 2012. Homework 2.

Due: Monday 7/May. 2012.

Question 1. Dynamic Programming.

Consider the distribution:

$$\pi(\vec{x}) = \frac{1}{Z} e^{\sum_{i=0}^{d-1} x_i x_{i+1} + \sum_{i=0}^{d} x_i y_i}. \tag{1}$$

Describe how dynamic programming can be used to solve the following two tasks: (a) to find the most probable state $\vec{x}^* = \arg\max_{\vec{x}} \pi(\vec{x})$, (b) to re-express the distribution in form $\pi(\vec{x}) = \pi_d(x_d)\pi_{d-1}(x_{d-1}|x_d)...\pi_0(x_0|x_1)$ and to compute the distributions $\pi_i(x_i|x_{i+1})$ $(i = 0, ..., d-1)$, $\pi_d(x_d)$, and the normalization constant Z.

Implement dynamic programming to solve both tasks on the example above. Set $d = 2$, $x_i \in \{\pm 1\}$, and $y_0 = 1.0, y_1 = 2.0, y_2 = -4.0$. After determining the distribution $\pi(\vec{x}) = \pi_d(x_d)\pi_{d-1}(x_{d-1}|x_d)...\pi_0(x_0|x_1)$, draw 10 samples from it.

Question 2. Kalman Filter.

Describe the basic Kalman filter for a one-dimensional problem with distributions $p(x_{t+1}|x_t) = N(\mu + x_t, \sigma_p^2)$, $p(y_t|x_t) = N(x_t, \sigma_m^2)$ ($N(.,.)$ is a normal distribution).

Implement the Kalman filter starting from an initial distribution $P(x_0) = N(0, \sigma_0^2)$. Run the algorithm till $t = 2$ to estimate $P(x_2|y_1, y_2)$.

Set $\sigma_0 = 0.5$, $y_1 = 1.2$, $y_2 = 2.1$.

Run the simulation for two different values of the parameters of the Kalman filter. First, set $\mu = 1.0$, $\sigma_p = 3.0$, and $\sigma_m = 0.1$. Second, set $\mu = 1.0$, $\sigma_p = 0.1$ and $\sigma_m = 3.0$. For each case, plot the graph of the distributions $p(x_0), p(x_1|x_0), p(x_1|y_1), p(x_2|y_1)$, $p(x_2|y_2, y_1)$. What differences do you see for the two cases?

Question 3. Particle Filters (Bootstrap).

Nonlinear filtering assumes a prediction (state equation) model $p(x_{t+1}|x_t)$ for the state variable x_t and an observation model $p(y_t|x_t)$ for the observation y_t. Describe how the state distribution $p(x_t|y_t, ..., y_1)$ can be expressed recursively in terms of the prediction model, the observation model, and the initial distribution $p(x_0)$.

Describe the particle filtering algorithm. Implement this algorithm for the following problem (discussed in class). The prediction is $p(x_{t+1}|x_t) = N(\mu + x_t, \sigma^2)$ with $\mu = 1.0$ and $\sigma = 1.5$. The probability of the observation $m_t, \vec{y_t} = (y_{t,1}, ..., y_{t,m})$ is given by:

$$p(\vec{y_t}|x_t) \propto (1 - p_d)\lambda + \sum_{k=1}^{m_t} p_d \frac{1}{\sqrt{2\pi}\tau} e^{-(y_{t,k}-x_t)^2/(2\tau^2)}. \tag{2}$$

The data is $m_1 = 2, y_{1,1} = 1.0, y_{1,2} = 0.5$, $m_2 = 2, y_{2,1} = 2.1, y_{2,2} = 2.5$. Set $\lambda = 1.0$, $p_d = 0.9$, $\tau = 1.0$.

Sample for two time-steps. Use 10 particles, initialized to take position x_0 to be $-0.3, -0.2, -0.15, -0.1, -0.05, 0.0, 0.05, 0.1, 0.15, 0.2$.

Question 4. Rosenbluth Method

Implement the Rosenbluth method to sample a self-avoiding walk (SAW) on a two-dimensional lattice. Sample $N = 20$ steps from this algorithm. Do this $M = 10$ times and calculate the weights for each sample. Then resample (with replacement), using the weights, to obtain unbiased samples of a SAW.