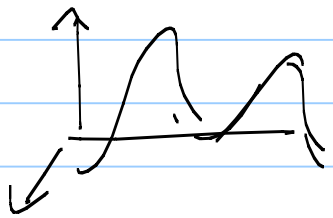
Lecture 6

Note Title 4/2/2006

Example: From Book.

$$f(x,y) = 0.5\, e^{-90(x-0.5)^2 - 45(y+0.1)^4} + e^{-45(x+0.4)^2 - 60(y-0.5)^2}$$

in domain $[-1,1] \times [-1,1]$



Two peaks $(0.5, -0.1)$ & $(-0.4, 0.5)$

Messy (multimodal) distribution & Truncated domain.

Approximate by $g(x,y)$ a truncated mixture of Gaussians centered on the two peaks.

$$g(x,y) \propto I_{(x,y)\in[-1,1]\times[-1,1]} \cdot \{(0.46)\, N(\underline{\mu}_1, \underline{\underline{\Sigma}}_1) + (0.54)\, N(\underline{\mu}_2, \underline{\underline{\Sigma}}_2)\}$$

$$\underline{\mu}_1 = (0.5, -0.1) \qquad \underline{\underline{\Sigma}}_1 = \begin{bmatrix} \frac{1}{180} & 0 \\ 0 & \frac{1}{90} \end{bmatrix}$$

$$\underline{\mu}_2 = (-0.4, 0.5) \qquad \underline{\underline{\Sigma}}_2 = \begin{bmatrix} \frac{1}{90} & 0 \\ 0 & \frac{1}{120} \end{bmatrix}$$

Sample as follows:

Draw sample from $N(\underline{\mu}_1, \underline{\underline{\Sigma}}_1)$ with prob 0.46 otherwise sample from $N(\underline{\mu}_2, \underline{\underline{\Sigma}}_2)$ (with prob 0.54.

Reject sample ($w = 0$) if it falls outside range $[-1,1] \times [-1,1]$.

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Importance Sampling can be super-efficient. The variance of $\bar{I}_m$ can be smaller than that obtained by sampling from $\pi(x)$

i.e. variance = 0 if $g(x) \propto \pi(x)h(x)$ and normalization constant known — but, if so, no need to sample.

The Rao-Blackwell-Gätwin method can be generalized to importance sampling.

Thm Let $f(z_1, z_2)$ & $g(z_1, z_2)$ be two distributions where the support of f is a subset of the support of g. Then.

$$\mathrm{Var}_g\left\{\frac{f(z_1,z_2)}{g(z_1,z_2)}\right\} \geq \mathrm{Var}_g\left\{\frac{f_1(z_1)}{g_1(z_1)}\right\}, \quad z$$

where $f_1(z_1) = \int f(z_1,z_2)\,dz_2$ & $g_1(z_1) = \int g(z_1,z_2)\,dz_2$ are marginals

Moral, in Monte Carlo computation you should do as much as possible analytically (e.g. marginalization.)

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Rule of Thumb for Importance Sampling.

Effective Sample Size

$$ESS(m) = \frac{m}{1 + var_g\{w(x)\}}$$

$w(x) = \frac{\pi(x)}{g(x)}$

If $\pi(x) = g(x)$, $ESS(m) = 1$.

Claim: $\frac{var_\pi\{h(x)\}}{var_g\{h(x)w(x)\}} \simeq \frac{1}{1 + var_g\{w(x)\}}$

only an approximation for $\pi(x) \approx g(x)$.

This claim, when true, implies that you can estimate the effectiveness of a sampling distribution $g(x)$ independent of $h(x)$. (Useful, but limited).

$ESS(m)$ gives a measure of how different the sampling distribution is from the target.

$var_g(w(x))$ can be measured by the coefficient of variation (c.o.v)

$$cv^2(w) = \frac{\sum_{j=1}^{m} (w_j - \bar{w})^2}{(m-1)\bar{w}^2}$$

$$\bar{w} = \frac{1}{m}\sum_{j=1}^{m} w_j$$

$w(x) = \frac{\pi(x)}{g(x)}$

Because if we only know $c\pi(x) = \pi(x)$ (unnormalized)

then $\frac{1}{(m-1)}\sum_{j=1}^{m} (w_j - \bar{w})^2 \sim c^2 Var_\pi(w)$

$\bar{w}^2 \sim c^2$

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Adaptive Importance Sampling.

It is good to try to learn as much as possible about the target distribution

Simple way, assume a t-distribution a trial distribution

$$g_0(x) = t_\alpha(x;\mu,\bar{\sigma}^2) = \frac{1}{(2\bar{\sigma}^2)^{1/2}} \frac{\Gamma((\alpha+1)/2)}{(\alpha\pi)^{1/2}\Gamma(\alpha/2)} \left\{1 + \frac{1}{2\bar{\sigma}^2}(y-\mu)^2\right\}^{-\frac{(\alpha+1)}{2}}$$

t-distribution falls off slower than the Gaussian.

Use weighted sampling to estimate the mean & covariance of the target distribution $\mu_1, \bar{\sigma}^2$

Then use a new trial

$$g_1(x) = t_\alpha(x, \mu_1, \bar{\sigma}_1^2)$$

For any statistic $\phi(x)$ (e.g. $\phi(x) = x, x^2, \ldots$)

Estimate ϕ for π by

$$\frac{1/m \sum_{i=1}^{m} \phi(x^i)\, w(x^i)}{\cdot\, 1/m \sum_{i=1}^{m} w(x^i)}.$$

Assumed that $h(x)$ is complicated function, want to evaluate it as rarely as possible.

Adaptive Importance Sampling.

Parametric form. $g_0(x)$ — need $g_1(x) = g(x, \lambda)$ for some λ

Pick $g_1(x)$ to minimize

$$\text{var}_{g_1}(w) = \sum_x \frac{\pi^2(x)}{g_1(x)} g_0(x) - 1.$$

as function of λ.

Estimate $\text{var}_{g_1}(w)$ by the coeff of variation

$$c\hat{v}^2(\lambda) = \hat{H}(\lambda) - 1, \quad \hat{H}(\lambda) = \frac{1}{m} \sum_{i=1}^{m} \frac{\pi^2(x^{(i)})}{g_1(x^{(i)}) g_0(x^{(i)})}$$

samples x^i from $g_0(x)$.

If normalization of $\pi(x)$ is unknown.

$$c\hat{v}^2(\lambda) = \frac{\hat{H}(\lambda)}{\bar{w}_0^2} - 1, \qquad \bar{w}_0 = \frac{1}{m} \sum_{i=1}^{m} \frac{\pi(x^{(i)})}{g_0(x^{(i)})}.$$

Problem: Adaptive methods risk being unstable....

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Justification of Claim:

$$\mathrm{var}_\pi\{h\}\{1+\mathrm{var}_g(\omega)\} \simeq \mathrm{var}_g\{h\omega\}$$

$g(x)$, $h(x)$ $\omega(x) = \frac{\pi(x)}{g(x)}$

$$\text{L.H.S} = \sum_x g(x)h^2(x)\left(\frac{\pi(x)}{g(x)}\right)^2 - \left(\sum_x g(x)h(x)\frac{\pi(x)}{g(x)}\right)^2$$

$$\text{L.H.S} = \sum_x \left(\frac{\pi(x)}{g(x)}\right)^2 h^2(x) - \left(\sum_x \pi(x)h(x)\right)^2.$$

$$\text{R.H.S} = \left(\sum_x \pi(x)h^2(x) - \left(\sum_x \pi(x)h(x)\right)^2\right) \times \left\{1 + \sum_x \frac{g(x)\pi^2(x)}{g^2(x)} - \left(\sum_x \frac{g(x)\pi(x)}{g(x)}\right)^2\right\}$$

$$\text{R.H.S} = \left(\sum_x \pi(x)h^2(x) - \left(\sum_x \pi(x)h(x)\right)^2\right)\left(\sum_x \frac{\pi^2(x)}{g(x)}\right).$$

For equality LHS = RHS is equivalent to

$$\sum_x \frac{\pi^2(x)h^2(x)}{g(x)} - \left(\sum_x \pi(x)h(x)\right)^2 \sum_x \frac{\pi^2(x)}{g(x)}$$

$$= \sum_x \pi(x)h(x)\left\{1 - \sum_x \frac{\pi^2(x)}{g(x)}\right\}$$

equality holds, if $\pi(x) = g(x)$.

Approximate holds if we do expansion in $g(x)/\pi(x)$. But can be violated easily