Swendsen - Wang.

Ising $X_i \in \langle -1, 1 \rangle$

Edges E.

Standard Metropolis is slow to converge when sampling from the Ising Model.

$$P(\underline{x}) = \frac{1}{Z} e^{\sum_{i,j \in E} X_i X_j / T}$$

particularly slow at low temperature.

Swendsen - Wang (SW) is a way to speed up the sampling by "dynamically grouping" sites into clusters.

Then the state of the entire cluster can be switched.

SW is also an example of Data Augmentation.

(*) Each subset V_i is connected.

Swendsen-Wang

First, Swendsen-Wang for the Potts model.

Each node s, can take colors $c_s \in \{1, 2 \ldots Q\}$

$$p(\underline{c}) = \frac{1}{Z} e^{\beta \sum_{s,t \in E} I(c_s = c_t)}$$

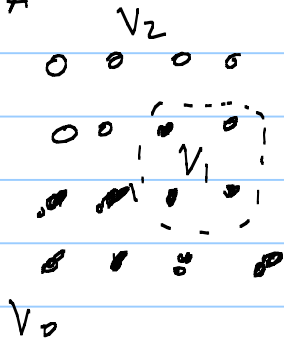$$\left\{ \begin{array}{l} I(c_s = c_t) = 1, \text{ if } c_s = c_t \\ = 0, \text{ otherwise} \end{array} \right\}$$

Ising model is a special case of Potts with $Q = 2$.

Notation: set of nodes V, partition into subsets $\{V_i : i = 1 \text{ to } n\}$ *

such that $\bigcup_{i=1}^{n} V_i = V$, $V_i \cap V_j = \phi$, $\forall i \neq j$.

All nodes in each subset have the same color.

State A

V_2

V_1

V_0

State B

V_2

V_1

V_0
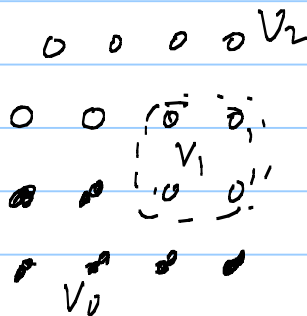
Partitions differ only by the state of the ⌐ ¬ regions.

$\Pi_A = (V_0 \cup V_1, V_2)$

$\Pi_B = (V_0, V_1 \cup V_2)$

SW Algorithm.

<u>1</u>. Define a new variable $V_{s,t}$ (binary-valued).

Set. $V_{s,t} = 0$ if $c_s \neq c_t$.

If $c_s = c_t$, set $V_{s,t} = 1$ with prob

$q_0 = 1 - e^{-\beta}$, (otherwise $V_{s,t} = 0$).

This yields a number of connected components (each is a subset of vertices of the same color).

2. Randomly select a connected component (e.g. the ⋯ region on previous page).

3. Randomly select a color for this connected component (uniform prob).

<u>Claim</u>: this algorithm is a Markov Chain Monte Carlo. ∴ satisfies detailed balance.

It encourages large groups at small temperature (large β)

Detailed Balance of SW.

Define:
$$C_A = C(V_0, V_1) = \{ (s,t) : s \in V_0, t \in V_1 \}$$
$$C_B = C(V_0, V_1) = \{ (s,t) : s \in V_0, t \in V_2 \}$$

These are the Swendsen-Wang cuts at π_A, π_B

Then, it follows that:

$$\frac{q(\pi_A \to \pi_B)}{q(\pi_B \to \pi_A)} = \frac{(1-q_0)^{|C_A|}}{(1-q_0)^{|C_B|}}$$

where $|C_A|, |C_B|$ are the sizes of C_A & C_B.

For the Potts model:
$$\frac{p(\pi_A)}{p(\pi_B)} = \frac{e^{-\beta |C_B|}}{e^{-\beta |C_A|}}.$$

Hence: if we set $q_0 = 1 - e^{-\beta}$,

then
$$\frac{q(\pi_B \to \pi_A)}{q(\pi_A \to \pi_B)} \frac{p(\pi_B)}{p(\pi_A)} = 1$$

which is detailed balance.

Alternative View of Swendsen-Wang.

on Potts model.

$$\pi(\underline{c}) \qquad \pi(\underline{v}|\underline{c}) = \prod_{s,t \in E} \pi(V_{s,t}|\underline{c}_s, \underline{c}_t)$$

with $\pi(V_{s,t}=1|\underline{c}_s, \underline{c}_t) = 0$, if $\underline{c}_s \neq \underline{c}_t$

$\pi(V_{s,t}=1|\underline{c}_s, \underline{c}_t) = 1-e^{-\beta}$, if $\underline{c}_s = \underline{c}_t$

Then do Data Augmentation:

Fix $\underline{c}$,
   sample from $\pi(\underline{v}|\underline{c})$ to get
                     connected components.

Then sample from $\pi(\underline{c}|\underline{v})$ to get the
   colour of the connected components.

It can be shown that $\pi(\underline{c}|\underline{v})$ is
a uniform distribution on each colour for
each connected component.
   (By definition, each connected component
   must have the same colour).

Extending Swendsen-Wang (Barbu & Zhu)

Standard SW is restricted to a few special distributions — e.g. Ising model & Potts model.

The problem is that detailed balance only occurs for these models.

But, we can extend Swendsen-Wang by Metropolis - Hastings.

Use SW to make proposals $q(\pi_A \to \pi_B)$, then accept proposals with probability

$$\alpha(\pi_A \to \pi_B) = \min\left(1, \frac{q(\pi_B \to \pi_A)}{q(\pi_A \to \pi_B)} \frac{p(\pi_B)}{p(\pi_A)}\right)$$

(Note: original SW is a special case where $\alpha(\pi_A \to \pi_B) = 1$, the proposal is automatically accepted because of detailed balance.)