Lecture 20.

Note Title 5/15/2006

Multiple-Try-Metropolis

(Enables bigger jumps than M-H in one step.)

There are many ways to extend Metropolis
→ here is one.

Define $\omega(\underline{x}, \underline{y}) = \pi(\underline{x})\, T(\underline{y} | \underline{x})\, \lambda(\underline{x}, \underline{y})$

$\lambda(\underline{x}, \underline{y})$ non-negative symmetric function of $\underline{x}, \underline{y}$.

Current State $\underline{x}^t$.

MTM — Draw k independent trial proposals $\underline{y}_1 \ldots \underline{y}_k$ from $T(\ | \underline{x})$

Compute $\omega(\underline{y}_j, \underline{x})$

- Select $\underline{y}$ from $(\underline{y}_1, \ldots, \underline{y}_k)$ with probability prop to $\omega(\underline{y}_j, \underline{x})$

Produce reference set $\underline{x}_1^*, \ldots, \underline{x}_{k-1}^*$ from $T(\underline{x} | \underline{y})$

$\underline{x}_k^* = \underline{x}$

- Accept $\underline{y}$ with prob

$$r_g = \min\left\{1, \frac{\omega(\underline{y}_1, \underline{x}) + \ldots + \omega(\underline{y}_k, \underline{x})}{\omega(\underline{x}_1^*, \underline{y}) + \ldots + \omega(\underline{x}_k^*, \underline{y})}\right\}$$

Special Case
$T(\underline{y}|\underline{x})$ symmetric.
$\lambda(\underline{x},\underline{y}) = T^{-1}(\underline{y}|\underline{x})$

r_g = generalized M-H ratio.

Same as M-H if $k=1$.

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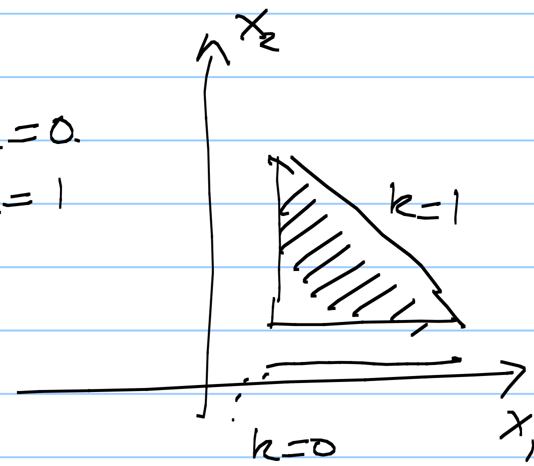
Note Title 5/12/2006

Reversible Jumps:

changing the dimension of the space.

Example

1 dim space $k=0$.
2 dim space $k=1$



states $\left\{ \begin{matrix} x, k=0 \\ (x_1, x_2), k=1 \end{matrix} \right\}$

π_0 - uniform in triangle
π_1 - uniform in $[0,1]$

$p \to$ prob that data lies in space $k=0$
$1-p \to$ prob that data lies in space $k=1$

Full distribution $p\pi_0 + (1-p)\pi_1$

To sample from this distribution is simple

with prob p, sample from π_0
$1-p$, sample from π_1

But to design an MCMC, we must be able to jump between space $k=0$ & space $k=1$

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Design three moves:

(a) For $k=0$, $x \to U(x-\epsilon, x+\epsilon)$ + boundary conditions at $x=0$, $x=1$

(b) For $k=1$, swap x_1 & x_2.

(c) Jump between $k=0$ & $k=1$ by

(i) for $k=0$, choose u from $U(0,1)$ and propose $(x_1, x_2) = (x, u)$

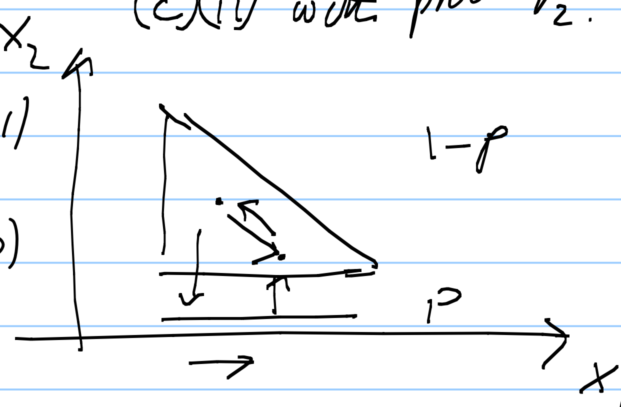
(ii) for $k=1$, set $x = x_1$

At each step:

For $k=0$, choose (a) with prob $1-r_1$,
(c)(i) with prob r_1

For $k=1$, choose (b) with prob $1-r_2$
(c)(ii) with prob r_2.

Move within triangle ($k=1$) along diagonal.
From triangle to segment ($k=0$)
Within segment.



More generally <u>Reversible Jumps</u>: (P. Green)

$\mathcal{X}$, $\mathcal{Y}$ is a subspace of $\mathcal{X}$ with lower dimension.

$\pi(\underline{x}) \propto \left(q_0(\underline{x}) \,|_{\underline{x} \in \mathcal{Y}} + q_1(\underline{x}) \right)$

Un-normalized probability distribution.

Need jumps $\mathcal{X} \to \mathcal{Y}$ & $\mathcal{Y} \to \mathcal{X}$.

Requires "matching space" $\mathcal{Z}$, so that $\mathcal{Y} \times \mathcal{Z}$ has same dimension as $\mathcal{X}$, and matching proposal.

$g(\underline{z}|\underline{y})$

Special Case: $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$

$\underline{x} = (\underline{y}, \underline{z})$, $\mathcal{Y} \sim \mathcal{Y} \times \{z_0\}$ for some $z_0 \in \mathcal{Z}$

To jump from $\mathcal{Y}$ to $\mathcal{X}$,

expansion: $\left\{ \begin{array}{l} \text{first propose } \underline{y} \to \underline{y}' \text{ from } T_1(\underline{y}'|\underline{y}) \\ \text{propose } \underline{z}' \text{ from } g(\underline{z}'|\underline{y}') \\ \text{let } \underline{x}' = (\underline{y}', \underline{z}') \end{array} \right.$

from $\mathcal{X}$ to $\mathcal{Y}$,

contraction: $\left\{ \begin{array}{l} \text{drop } \underline{z} \text{ component of } \underline{x} \\ \text{propose } \underline{y}' \text{ from } T_2(\underline{y}'|\underline{y}) \end{array} \right.$

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Accept, expansion proposal $\underline{y} \to \underline{x}'$ with probability

$$\alpha = \min\left\{1, \frac{q_1(\underline{y}', \underline{z}')\, T_2(\underline{y}\,|\,\underline{y}')}{q_0(\underline{y})\, T_1(\underline{y}'\,|\,\underline{y})\, g(\underline{z}'\,|\,\underline{y}')}\right\}$$

Accept, contraction proposal $\underline{x} \to \underline{y}'$ with prob.

$$\beta = \min\left\{1, \frac{q_0(\underline{y}')\, T_1(\underline{y}\,|\,\underline{y}')\, g(\underline{z}\,|\,\underline{y})}{q_1(\underline{y}, \underline{z})\, T_2(\underline{y}'\,|\,\underline{y})}\right\}$$

Expansion and Contraction together satisfy detailed balance.

The Expansion Proposal involves first "proposing" and then "lifting" (e.g. increase dimension of space)

Alternatively you can "lift" first and then "propose".

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More precisely,

to get proposal $\underline{y} \to \underline{x}'$

draw $\underline{z} \sim g(\underline{z}|\underline{y})$ and then draw

$\underline{x}'$ from $S_1[\underline{x} \cdot | (\underline{y}, \underline{z})]$

Contraction, propose $\underline{x} \to \underline{x}' = (\underline{y}', \underline{z}')$

from $S_2(\underline{x}'|\underline{x})$

then drop $\underline{z}'$.

Acceptance probabilities are:

$$\alpha' = \min\left\{1, \frac{q_1(\underline{x}')\, S_2((\underline{y}, \underline{z}) | \underline{x}')}{q_0(\underline{y})\, g(\underline{z}|\underline{y})\, S_1(\underline{x}' | (\underline{y}, \underline{z}))}\right\}$$

$$\beta' = \min\left\{1, \frac{q_0(\underline{y}')\, g(\underline{z}'|\underline{y}')\, S_1(\underline{x} | (\underline{y}', \underline{z}'))}{q_1(\underline{x})\, S_2((\underline{y}', \underline{z}') | \underline{x})}\right\}$$

Both S_1 & S_2 are proposals in the higher dimensional space $\mathcal{X}$,

but T_1 and T_2 are proposeds in lower-dim space $\mathcal{Y}$.

Other things being equal, prefer lifting after proposing...