Hidden Markov Models (HMM's)

Note Title 5/18/2008

Previous lectures assume that the samples are i.i.d (independently identically distributed).

But data often appears in sequences. There is dependence (stochastic) between different elements of the sequence.

Discrete Markov Processes

N - distinct states $S_1, \ldots, S_N$.

state at time t is q_t $\quad q_t = S_i$ means system in state S_i.

$P(q_{t+1} = S_j \mid q_t = S_i, q_{t-1} = S_k, \ldots)$

first-order Markov model.

$P(q_{t+1} = S_j \mid q_t = S_i, q_{t-1} = S_k, \ldots) = P(q_{t+1} = S_j \mid q_t = S_i)$

the future is independent of the past, except for the preceding time state.

Continue with First-order Markov model.

transition probabilities $a_{ij} = P(q_{t+1} = S_j \mid q_t = S_i)$

$a_{ij} \geq 0$ & $\sum_{j=1}^{N} a_{ij} = 1$, for all i.

transition prob. is independent of time.

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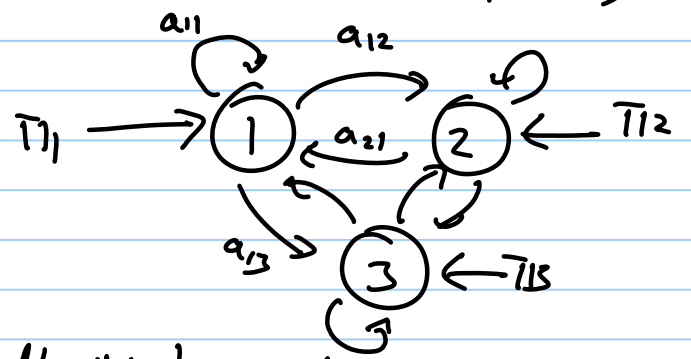
Initial probability. $\pi_i \equiv P(q_i = S_i)$ $\quad \sum_{i=1}^{} \pi_i = 1$



In an observable Markov model, we can directly observe the states $\{q_t\}$.
(This enables us to learn the transition probs)

Observation sequence $O = Q = \{q_1 \dots q_T\}$

$$P(O = Q | A, \pi) = p(q_1) \prod_{t=2}^{T} p(q_t | q_{t-1}) = \pi_{q_1} a_{q_1 q_2} \dots a_{q_{T-1} q_T}.$$

Example: Urns with 3 types of ball
S_1 = red, S_2 = blue, S_3 = green

// state: the urn we draw the ball from.

Initial prob $\pi = [0.5, 0.2, 0.3]$

Transition. a_{ij} $\quad A = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$

sequence $O = (S_1, S_1, S_3, S_3)$

$$P(O | A, \pi) = P(S_1) P(S_1 | S_1) P(S_3 | S_1) P(S_3 | S_3)$$
$$= \pi_1 \cdot a_{11} \cdot a_{13} \cdot a_{33} = (0.5)(0.4)(0.3)(0.8)$$
$$= 0.048.$$

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Learning Parameters for HMM.

Suppose we have K sequences of length T.

q_t^k is state at time t of k^{th} sequence.

$$\hat{\pi}_i = \frac{\#[\text{sequence starting with } S_i]}{\#[\text{number of sequences}]} = \frac{\sum_k I(q_1^k = S_i)}{K}$$

$$\hat{a}_{ij} = \frac{\#[\text{transitions from } S_i \text{ to } S_j]}{\#[\text{transitions from } S_i]}$$

$$= \frac{\sum_k \sum_{t=1}^{T-1} I\,(q_t^k = S_i \text{ and } q_{t+1}^k = S_j)}{\sum_k \sum_{t=1}^{T-1} I(q_t^k = S_i)}$$

E.G. $\hat{a}_{12}$ is no. of times a blue ball follows a red ball divided by the total no. of red balls.

Note: These learning formula are intuitive. But it is important to realize that they are obtain by ML (max likelihood)

$$\hat{A}, \hat{\pi} = \text{ARG MAX} \prod_{k=1}^{K} P(O = Q_k \mid A, \pi)$$

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Hidden Markov Models HMM's

States are not directly observable, but we have an observation from each state.

States $q_t \in \{S_1 \ldots S_N\}$

observable $O_t \in \{v_1 \ldots v_M\}$

$b_j(m) \equiv P(O_t = v_m \mid q_t = S_j)$

observation probability that we observe v_m if the state is S_j.

I.e. two sources of stochasticity

the observation is stocastic. $b_j(m)$

the transition is stochastic a_{ij}.

Back to the urn analogy:

Let the urn contain balls with different colours. → E.G. Urn 1 mostly red

Urn 2 mostly blue

Urn 3 mostly green.

The observation is the ball colour, but we don't know which urn it comes from (the state)

(5) # HMM. Formalism.

Elements:

1. N: No. of states
   $S = \{S_1 \ldots S_N\}$
2. M: No of observation symbols in alphabet
   $V = \{v_1, v_2 \ldots v_M\}$
3. State transition probabilities:
   $A = [a_{ij}]$ $\quad a_{ij} = P(q_{t+1} = S_j \mid q_t = S_i)$
4. Observation probabilities:
   $B = [b_j(m)]$, $\quad b_j(m) = P(O_t = v_m \mid q_t = S_j)$
5. Initial state probabilities:
   $\Pi = [\pi_i]$ where $\pi_i = P(q_i = S_i)$.

$\lambda = (A, B, \Pi)$ specifies the parameter set of an HMM.

Three Basic Problems

(1.) Given a model λ, evaluate the prob $P(O \mid \lambda)$ of any sequence $O = (O_1, O_2 \ldots O_T)$

(2) Given a model and observation sequence O, find state sequence $Q = \{q_1, q_2 \ldots q_T\}$, which has highest probability of generating O
find Q^* s.t. $Q^* = \arg\max P(Q \mid O, \lambda)$

(3) Given training set of sequences $\chi = \{O^{(k)}\}_k$
find $\lambda^* = \arg\max P(\chi \mid \lambda)$.

(6) HMM's.

Problem 1. Evaluation.

Given an observation $O = \{O_1 O_2 \dots O_T\}$ and a state sequence $Q = \{q_1 q_2 \dots q_T\}$, the prob. of observing O given Q is

$$P(O|Q,\lambda) = \prod_{t=1}^{T} P(O_t|q_t,\lambda) = b_{q_1}(O_1) \cdot b_{q_2}(O_2) \dots b_{q_T}(O_T)$$

But we don't know Q.

The prior prob. of the state sequence is

$$P(Q|\lambda) = P(q_1) \prod_{t=2}^{T} P(q_t|q_{t-1}) = \pi_{q_1} a_{q_1 q_2} \dots a_{q_{T-1} q_T}$$

Joint probability is

$$P(O,Q|\lambda) = P(q_1) \prod_{t=2}^{T} P(q_t|q_{t-1}) \prod_{t=1}^{T} P(O_t|q_t)$$

$$= \pi_{q_1} b_{q_1}(O_1) a_{q_1 q_2} b_{q_2}(O_2) \dots a_{q_{T-1} q_T} b_{q_T}(O_T)$$

We can compute

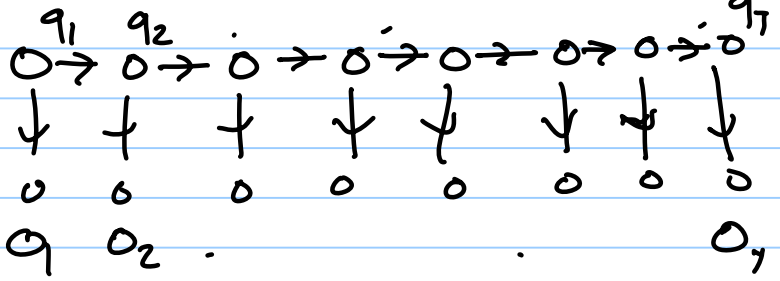
$$P(O|\lambda) = \sum_{\text{all possible } Q} P(O,Q|\lambda)$$

But this summation is impractical directly, because there are too many possible Q. N^T.

(7) # HMM's

But there is an efficient procedure to calculate $P(O|\lambda)$ called the forward-backward procedure (essentially – dynamic programming). This exploits the structure of the distribution

Divide the sequence into parts $t=1$ to t, $t+1$ to T.



Forward variable $\alpha_t(i)$ is prob. of observing the partial sequence $\{O_1 \ldots O_t\}$ and being in state S_t at time t, (given the model λ)

$$\alpha_t(i) \equiv P(O_1 \ldots O_t, q_t = S_i | \lambda)$$

This can be computed recursively:

Initialization → $\alpha_1(i) = P(O_1, q_1 = S_i|\lambda)$
$= P(O_1|q_1 = S_i, \lambda) P(q_1 = S_i|\lambda)$
$= \pi_i b_i(O_1)$

recursion $\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(O_{t+1})$

see book for details

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HMM's

Intuition: $\alpha_t(i)$ explains first t observations and ends in state S_i

multiply by prob a_{ij} to get to state S_j at $t+1$
multiply by prob of generating $(t+1)^{th}$ observation $b_j(O_{t+1})$
then sum over all possible states S_i at time t.

Finally, $P(O|\lambda) = \sum_{i=1}^{N} P(O, q_T = S_i | \lambda)$

$= \sum_{i=1}^{N} \alpha_T(i)$ //

Computing $\alpha_t(i)$ is $O(N^2 T)$

This solves the first problem — computing the prob of generating the data given the model.

An alternative algorithm (which we need later) is backward variable $\beta_t(i)$

$$\beta_t(i) \equiv P(O_{t+1}, \ldots, O_T | q_t = S_i, \lambda)$$

Initialize $\beta_T(i) = 1$

recursion. $\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(O_{t+1}) \beta_{t+1}(j)$

HMM's

Finding the state sequence. – 2nd Problem

Again exploit the linear structure.

Greedy

Define $\gamma_t(i)$ as prob of state S_i at time t given O and λ.

$$\gamma_t(i) = P(q_t = S_t \mid O, \lambda)$$
$$= \frac{P(O \mid q_t = S_i, \lambda)\, P(q_t = S_i \mid \lambda)}{P(O \mid \lambda)}$$
$$= \frac{\alpha_t(i)\,\beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j)\,\beta_t(j)} \quad //$$

normalization ensures that $\sum_i \gamma_t(i) = 1$.

$\alpha_t(i)$ forward variable explains the starting part of the sequence until time t ending in S_i, backward variable $\beta_t(i)$ explains the remaining part of the sequence up to time T.

We can try to estimate the state by choosing $q_t^* = \arg\max_i \gamma_t(i)$ for each t.

But, this ignores the relations between neighboring states. It may be inconsistent $q_t^* = S_i$, $q_{t+1}^* = S_j$ but $a_{ij} = 0$

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HMM's

Viterbi Algorithm (Dynamic Programming)

Define $\delta_t(i)$ is the prob of the highest probability path that accounts for all the first t observations and ends in S_i:

$$\delta_t(i) = \max_{q_1 \cdots q_{t-1}} P(q_1, q_2 \ldots, q_{t-1}, q_t = S_i, O_1 \ldots O_t | \lambda)$$

calculate this recursively:

1. Initialize: $\delta_1(i) = \pi_i b_i(O_1)$, $\psi_1(i) = 0$.

2. Recursion. $\delta_t(j) = \max_i \delta_{t-1}(i) a_{ij} b_j(O_t)$

$$\psi_t(j) = \arg\max_i \delta_{t-1}(i) a_{ij}.$$

3 Termination:

$$p^* = \max_i \delta_T(i)$$

$$q_T^* = \arg\max_i \delta_T(i)$$

4. Path (state sequence) backtracking:

$$q_t^* = \psi_{t+1}(q_{t+1}^*), \quad t = T-1, T-2 \ldots 1$$

Intuition $\psi_t(j)$ keeps track of the state that maximizes $\delta_t(j)$ at time $t-1$

Same complexity $O(N^2 T)$.

HMM's

Learning Model Parameters. Problem 3:

$\chi = \{O^{(k)}\}_{k=1}^{K}$ set of sequences.

$$P(\chi|\lambda) = \prod_{k=1}^{K} P(O^{k}|\lambda)$$

$$\lambda^{*} = \underset{\lambda}{\text{ARG MAX}}\ P(\chi|\lambda).$$

This is performed by a combination of EM and dynamic programming.

Defn: $\xi_t(i,j)$ prob. of being in state S_i at time t and in state S_j at $t+1$ given observation O and λ:

$$\xi_t(i,j) = P(q_t = S_i, q_{t+1} = S_j \mid O, \lambda)$$

$$\xi_t(i,j) = \frac{\alpha_t(i)\, a_{ij}\, b_j(O_{t+1})\, \beta_{t+1}(j)}{\sum_k \sum_l \alpha_t(k)\, a_{kl}\, b_l(O_{t+1})\, \beta_{t+1}(l)}$$

Note: if Markov model is observable, then both $\gamma_t(i)$ & $\xi_t(i,j)$ are 0/1

HMM's

Baum-Welch algorithm → EM.

At each iteration,

E-step compute $\xi_t(i,j)$ & $\gamma_t(i)$ given current $\lambda = (A, B, \Pi)$

M-step recalculate λ given $\xi_t(i,j)$ & $\gamma_t(i)$

Alternate the two steps until convergence.

Indicator variables z_i^t

$$z_i^t = \begin{cases} 1, & \text{if } q_t = S_i \\ 0, & \text{otherwise} \end{cases}$$

and $$z_{ij}^t = \begin{cases} 1, & \text{if } q_t = S_i \text{ & } q_{t+1} = S_j \\ 0, & \text{otherwise.} \end{cases}$$

(Note, these are 0/1 in case of observable Markov model)

Estimate them in the E-step as

$$E[z_i^t] = \gamma_t(i)$$
$$E[z_{ij}^t] = \xi_t(i,j)$$

In M-step, count the expected no. of transitions from S_i to S_j $\sum_t \xi_t(i,j)$ and total no. of transitions from S_i is $\sum_t \gamma_t(i)$

HMM's

This gives transition probabilities from S_i to S_j

$$\hat{a}_{ij} = \sum_{t=1}^{T-1} \xi_t(i,j) \Big/ \sum_{t=1}^{T-1} \gamma_t(i)$$

(soft counts instead of real counts.)

$$\hat{b}_j(m) = \frac{\sum_{t=1}^{T} \gamma_t(j)\, I(O_t = v_m)}{\sum_{t=1}^{T} \gamma_t(j)}$$

For multiple observation sequences.

$$\chi = \{O^k\}_{k=1}^{K}$$

$$P(\chi|\lambda) = \prod_{k=1}^{K} P(O^k|\lambda).$$

$$\hat{a}_{ij} = \frac{\sum_{k=1}^{K} \sum_{t=1}^{T_k-1} \xi_t^k(i,j)}{\sum_{k=1}^{K} \sum_{t=1}^{T_k-1} \gamma_t^k(i)}$$

$$\hat{b}_j(m) = \frac{\sum_{k=1}^{K} \sum_{t=1}^{T_k-1} \gamma_t^k(j)\, I(O_t^k = v_m)}{\sum_{k=1}^{K} \sum_{t=1}^{T_k-1} \gamma_t^k(j)}$$

$$\hat{\pi}_i = \sum_{k=1}^{K} \gamma_1^k(i) \Big/ K$$

HMM's

Recap:

We have given algorithms to solve the three problem:

(1) compute $P(O|\lambda)$

(2) compute $Q^* = \arg\max_Q P(Q|O,\lambda)$

(3.) compute $\lambda^* = \arg\max_\lambda P(\lambda|\lambda)$

$P(O|\lambda)$ is used for model selection

Suppose we have two alternative models for the data $P(O|\lambda_1)$ $P(O|\lambda_2)$

select model 1 if $P(O|\lambda_1) > P(O|\lambda_2)$

model 2 otherwise.

i.e. detect which model generated the sequences.

Do this for multiple models with training data for each.

$\lambda_1^*, \ldots \lambda_w^* = \arg\max_\lambda P(x^1|\lambda), P(x^2|\lambda) \ldots P(x^w|\lambda)$

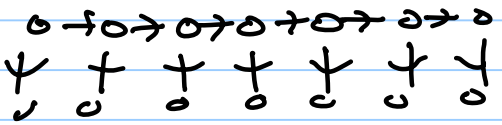
Model selection.

Use this to build speech recognition system.

HMM's

Further extensions of HMM's are described in the book.

The basic idea is to exploit the one-dimensional structure of the model.



Enables dynamic programming to perform rapid computation.

EM - algorithm for learning the model parameters

Multiple models - model selection.