Note Title 3/29/2008

Linear Discriminant Analysis (LDA)

Start with 2 classes

$$z = \underline{w}^T \underline{x}.$$

$\underline{m}_1$, m_1 are the means of samples from C_1 before and after projection.

Sample $\mathcal{X} = \{\underline{x}^t, \underline{r}^t\}$ $\quad r^t = 1$, if $\underline{x}^t \in C_1$

$\quad\quad = 0$, if $\underline{x}^t \in C_2$.

$$m_1 = \frac{\sum_t \underline{w}^t \underline{x}^t r^t}{\sum_t r^t} = \underline{w}^T \underline{m}_1$$

$$m_2 = \frac{\sum_t \underline{w}^T \underline{x}_t (1-r^t)}{\sum_t (1-r^t)} = \underline{w}^T \underline{m}_2.$$

The scatter of samples from C_1 & C_2 after projection

$$s_1^2 = \sum_t (\underline{w}^T \underline{x}^t - m_1)^2 r^t$$

$$s_2^2 = \sum_t (\underline{w}^T \underline{x}^t - m_2)^2 (1-r^t)$$

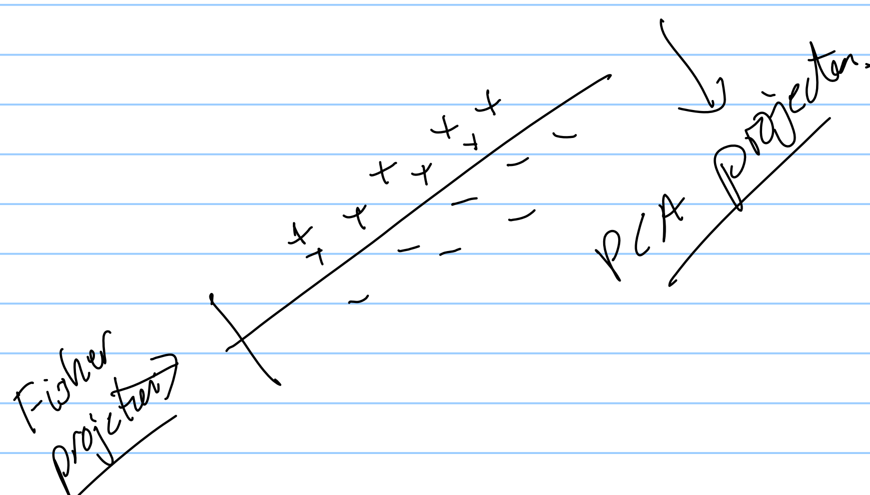
Fisher's Linear Discriminant maximizes

$$J(\underline{w}) = \frac{(m_1-m_2)^2}{s_1^2+s_2^2}$$

Rewrite: $(m_1-m_2)^2 = (\underline{w}^T \underline{m}_1 - \underline{w}^T \underline{m}_2)^2 = \underline{w}^T \underline{\underline{S}}_B \underline{w}$

where $\underline{\underline{S}}_B = (\underline{m}_1 - \underline{m}_2)(\underline{m}_1 - \underline{m}_2)^T$

between-class scatter matrix



(2) (LDA) (cont)

$$s_1^2 = \sum_t (\underline{w}^T \underline{x}^t - m_1)^2 r^t$$
$$= \sum_t \underline{w}^T (\underline{x}^t - \underline{m}_1)(\underline{x}^t - \underline{m}_1)^T \underline{w}\, r^t$$
$$= \underline{w}^T \underline{\underline{S}}_1 \underline{w}$$

where $\underline{\underline{S}}_1 = \sum_t r^t (\underline{x}^t - \underline{m}_1)(\underline{x}^t - \underline{m}_1)^T$

within-class scatter matrix for C_1.

$$s_1^2 + s_2^2 = \underline{w}^T \underline{\underline{S}}_w \underline{w} \qquad \underline{\underline{S}}_w = \underline{\underline{S}}_1 + \underline{\underline{S}}_2$$

$$J(\underline{w}) = \frac{\underline{w}^T \underline{\underline{S}}_B \underline{w}}{\underline{w}^T \underline{\underline{S}}_w \underline{w}} = \frac{|\underline{w}^T (\underline{m}_1 - \underline{m}_2)|^2}{\underline{w}^T \underline{\underline{S}}_w \underline{w}}$$

Formulate as

$$\underline{w}^T \underline{\underline{S}}_B \underline{w} - \underbrace{\lambda}_{\text{Lagrange multiplier}} (\underline{w}^T \underline{\underline{S}}_w \underline{w} - 1)$$

$\frac{\partial}{\partial \underline{w}}$

$$\underline{\underline{S}}_B \underline{w} = \lambda \underline{\underline{S}}_w \underline{w}$$

$$\underline{\underline{S}}_B \underline{w} = (\underline{m}_1 - \underline{m}_2)\underbrace{(\underline{m}_1 - \underline{m}_2)^T \underline{w}}$$
$$\propto (\underline{m}_1 - \underline{m}_2)$$

Hence solution $\underline{w} \propto \underline{\underline{S}}_w^{-1} (\underline{m}_1 - \underline{m}_2)$

magnitude of $\underline{w}$ is irrelevant.
just need the direction.

(3) LDA, for $K > 2$.

Task: find $\underline{\underline{w}}$ s.t. $\underline{z} = \underline{\underline{w}}^T \underline{x}$

where $\underline{z}$ is k-dim & $\underline{\underline{w}}$ is $d \times k$.

within-class scatter for C_i

$$\underline{\underline{S}}_i = \sum_t r_i^t (\underline{x}^t - \underline{m}_i)(\underline{x}^t - \underline{m}_i)^T$$

$r_i^t = 1$, if $\underline{x}^t \in C_i$

$= 0$, otherwise.

Total within-class scatter is

$$\underline{\underline{S}}_W = \sum_{i=1}^{K} \underline{\underline{S}}_i$$

Scatter of means — $\underline{m} = \frac{1}{K} \sum_{i=1}^{K} \underline{m}_i$

$$\underline{\underline{S}}_B = \sum_{i=1}^{K} N (\underline{m}_i - \underline{m})(\underline{m}_i - \underline{m})^T \qquad N_i = \sum_t r_i^t.$$

After projection:

Both $k \times k$ matrices $\left\{ \begin{array}{ll} \text{between-class scatter} & \underline{w}^T \underline{\underline{S}}_B \underline{w} \\ \text{within-class scatter} & \underline{\underline{w}}^T \underline{\underline{S}}_W \underline{\underline{w}} \end{array} \right.$

$$J(\underline{w}) = \frac{| \underline{\underline{w}}^T \underline{\underline{S}}_B \underline{\underline{w}} |}{| \underline{w}^T \underline{\underline{S}}_W \underline{\underline{w}} |}$$

solution $\rightarrow$ $\underline{\underline{w}}$'s columns are the largest eigenvectors of $\underline{\underline{S}}_W^{-1} \underline{\underline{S}}_B$.

$\underline{\underline{S}}_B$ has max-rank $K-1$.

Lower-dim space dim $K-1$