YANG–MILLS INSTANTONS IN SELF-DUAL SPACETIMES

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The zero modes of a Yang–Mills instanton in a self-dual spacetime are related to the solutions of the zero mass Dirac equation in the adjoint representation. The index theorem is used to determine the number of parameters of instantons in such spacetimes. The result is illustrated by a calculation in Taub–NUT spacetime using the instanton constructed by the Charap and Duff prescription.

In a recent paper [1] Charap and Duff have given a prescription for constructing an SU(2) Yang–Mills instanton in any Ricci-flat spacetime. In the present paper we will assume the existence of an instanton of arbitrary group G and Pontryagin integer number n and derive an expression for the number of parameters characterizing it in a self-dual spacetime. Recent work by Hawking and Pope [2] suggest such spacetimes may be important in a quantum theory of gravity.

In the adjoint representation the zero mass Dirac equation is written as

$$\gamma^{\mu}(\nabla_{\mu}\delta_{ij} - C_{ijk}A^{0k}{}_{\mu})\,\lambda^{j} = 0\,, \tag{1}$$

where λ^{jl} is a vector of Dirac 4-spinors, the C_{ijk} are the structure constants of the Lie group and A^{k}_{μ} is the instanton field.

This can be written in the 2-component spinor form as

$$(\nabla_{AA'}\delta_{ij} - C_{ijk}A^{k}_{AA'})\lambda^{jA} = 0\,, \tag{2}$$

where λ^{jA} is the positive helicity part of λ^{j}. There is a similar equation for $\widetilde{\lambda}^{jA'}$, the negative helicity part, but this will have no normalizable solutions. One way of seeing this is by considering the curvature components of the Yang–Mills and gravitational fields. These can be split up into self-dual and anti-self-dual parts which are Φ^{i}_{AB} and $\widetilde{\Phi}^{i}_{A'B'}$ for the Yang–Mills and Ψ_{ABCD} and $\widetilde{\Psi}^{i}_{A'B'C'D'}$ for the gravitational field. The self-duality of the instanton and the spacetime will imply $\widetilde{\Phi}^{i}_{A'B'} = \widetilde{\Psi}_{A'B'C'D'} = 0$ while Φ^{i}_{AB} and Ψ_{ABCD} will have some non-zero components. Thus only the positive helicity states "see" the fields and so there are only positive helicity bound states.

We can operate on the left-hand side of eq. (1) by $\gamma^{\nu}(\nabla_{\nu}\delta_{li} - C_{lim}A^{m}_{\nu})$ and express the result in spinor form as

$$(\nabla^{BB'}\delta_{li} + C_{lmi}A^{mBB'})(\nabla_{BB'}\delta_{ij} + C_{ikj}A^{k}_{BB'})\lambda^{j}_{A}$$

$$= -C_{lmj}\,\Phi^{m}_{AB}\lambda^{kB}\,,$$

where $\Phi^{m}_{AB} = \frac{1}{2}F^{m}_{ABC'}{}^{C'}$ and $F^{m} = \mathrm{d}A^{m} + \frac{1}{2}C_{mpq} \times A^{p}\wedge A^{q}$.

In a self-dual spacetime $\widetilde{\Psi}_{A'B'C'D'} = 0$ which implies the existence of a pair of covariantly constant primed spinors. These satisfy the equations

$$\nabla_{BB'}\bar{\alpha}^{A'} = 0\,. \tag{4}$$

Now the zero mass perturbations $B^{i}_{AA'}$, of the instanton obey the equation

$$(\delta_{ik}\nabla^{BB'} + C_{ijk}A^{jBB'})(\delta_{kl}\nabla_{BB'} + C_{kjl}A^{j}_{BB'})B^{l}_{AA'}$$

$$= -C_{ijk}\Phi^{jB}_{A}B^{k}_{BA'}\,, \tag{5}$$

and can be chosen to satisfy the gauge condition

$$(\nabla_{AA'}\delta_{ij} - C_{ijk}A^{k}_{AA'})B^{iAA'} = 0\,. \tag{6}$$

Hence if we have λ^{i}_{A} satisfying eqs. (2) and (3) and $\bar{\alpha}_{A'}$ satisfying eq. (4), their product $B^{i}_{AA'} = \lambda^{i}_{A}\bar{\alpha}_{A'}$ will satisfy eqs. (5) and (6). Converseley if $B^{i}_{AA'}$ satisfies eqs. (5) and (6) and $\bar{\alpha}^{A'}$ satisfies eq. (4) then

$\mu^i_A = B^i_{AA'}\bar{\alpha}^{A'}$ will obey eqs. (1) and (2). Thus the number of zero mass perturbations of the instanton is twice the number of solutions of the Dirac equation in the adjoint representation.

In a curved spacetime with boundary the Index Theorem for the zero mass Dirac equation in the adjoint representation generalizes to

$$n_+ - n_- = 2\,C(G)\left\{-\frac{1}{384\pi^2}\int_M R^*_{\mu\nu o\rho}R^{\mu\nu o\rho}\sqrt{g}\,\mathrm{d}^4x + \frac{1}{16\pi^2}\int_M F^{i*}_{\mu\nu}F^{i\mu\nu}\sqrt{g}\,\mathrm{d}^4x - \tfrac{1}{2}\eta(0) + \int_{\partial M} Q\right\}, \quad (7)$$

where $^*R_{abcd} = \frac{1}{2}\epsilon_{abef}R^{ef}{}_{cd}\sqrt{g}$, $^*F^i_{ab} = \frac{1}{2}\epsilon_{abcd} \times F^{cd}\sqrt{g}$, n_+, n_- are the number of positive and negative helicity zero modes, respectively, and $C(G)$ is the Casimir operator of G. $\eta(s) = \Sigma_\lambda (\mathrm{sign}\,\lambda)|\lambda|^{-s}$, where the λ's are the eigenvalues of the Dirac operator on the boundary ∂M. The integrand Q contains terms involving the curvature, the second fundamental form of the boundary in M and the Yang–Mills potential.

Thus with $n_- = 0$ because of the self-duality of the gravitational and Yang–Mills fields the number of parameters is

$$4C(G)\left\{-\frac{1}{384\pi^2}\int_M R^*_{\mu\nu o\rho}R^{\mu\nu o\rho}\sqrt{g}\mathrm{d}^4x + \frac{1}{16\pi^2}\int_M F^{i*}_{\mu\nu}F^{i\mu\nu}\sqrt{g}\,\mathrm{d}^4x - \tfrac{1}{2}\eta(0) + \int_{\partial M} Q\right\}. \quad (8)$$

It is unclear exactly how many of these solutions correspond to infinitesimal gauge rotation. For the unique compact self-dual manifold K.3 these will be of form $D^\mu\Lambda^i$ and the gauge condition $D_\mu D^\mu\Lambda^i = 0$ will mean none exist. For non compact manifolds it seems reasonable to assume that there will be $3m$ where m is the number of distinct subgroups of SU(2) in G. This agrees with known results in Schwarzschild, Taub–NUT and flat spacetimes.

For the special case of flat spacetime and SU(2) this reduces to the well known formula $8n - 3$, where n is the Pontryagin number of the instanton.

We will now check the result by an explicit calculation in Taub–NUT spacetime with the SU(2) instanton constructed by Pope and Yuille [3] using the Charap and Duff prescription.

The self-dual Taub–Nut metric is [4]

$$\mathrm{d}s^2 = \left(\frac{r+M}{r-M}\right)\mathrm{d}r^2 + 4M^2\left(\frac{r-M}{r+M}\right)(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2 + (r^2 - M^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \quad (9)$$

with $M \leqslant r < \infty$, $0 \leqslant \psi \leqslant 4\pi$, $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$.

The instantion solution is

$$A^3 = \left\{1 - \frac{4M^2}{(r+M)^2}\right\}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi),$$

$$A^1 = -\left(\frac{r-M}{r+M}\right)(\sin\theta\sin\psi\,\mathrm{d}\phi + \cos\psi\,\mathrm{d}\theta),$$

$$A^2 = -\left(\frac{r-M}{r+M}\right)(\sin\theta\cos\psi\,\mathrm{d}\phi - \sin\psi\,\mathrm{d}\theta). \quad (10)$$

It follows from the Charap and Duff prescription that when there are no boundary terms the Pontryagin number P_{YM} of the instanton is given by

$$P_{\mathrm{YM}} = \tfrac{1}{2}\chi + \tfrac{1}{4}P_G, \quad (11)$$

where χ is the Euler number and P_G the volume integral contribution to the Pontryagin number of the manifold. For Taub–NUT $\chi = 1$, $P_G = 2$ and hence $P_{\mathrm{YM}} = 1$.

For Taub–NUT

$$\frac{1}{384\pi^2}\int_M R^*_{\mu\nu o\rho}R^{\mu\nu o\rho}\sqrt{g}\,\mathrm{d}^4x = \tfrac{1}{12},$$

$$\eta(0) = -1/6, \quad \int_{\partial M} Q = 0. \quad (12)$$

So the Index Theorem predicts $n_+ - n_- = 4$ but $n_- = 0$ by the previous argument so $n_+ = 4$. We solve the equations in the Kinnersley tetrad

$$l = \frac{16M^3}{(r+M)^2}\frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{8M\mathrm{i}}{\rho^2}\frac{\partial}{\partial\psi},$$

$$-2n = \frac{M}{(r+M)^2}\rho\frac{\partial}{\partial\rho} + \frac{\mathrm{i}}{2M}\frac{\partial}{\partial\psi}, \quad (13)$$

$$\sqrt{2}m = \frac{1}{r-M}\left\{-\mathrm{i}\cot\theta\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\theta}+\mathrm{i}\,\mathrm{cosec}\,\theta\frac{\partial}{\partial\phi}\right\},$$

$$\sqrt{2}\widetilde{m} = \frac{1}{r+M}\left\{\mathrm{i}\cot\theta\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\theta}-\mathrm{i}\,\mathrm{cosec}\,\theta\frac{\partial}{\partial\phi}\right\},$$

with $\rho^2 = 16M^2[(r-M)/(r+M)]$ and a dyad basis (o^A, ι^A) with $l^a = o^A\bar{o}^{A'}$, $n^a = \iota^A\bar{\iota}^{A'}$, $m^a = o^A\bar{\iota}^{A'}$, $\widetilde{m}^a = \iota^A\bar{o}^{A'}$.

With respect to these bases the solutions of eq. (2) are found to be

$$\lambda_A^3 = \frac{\mathrm{i}}{2}\frac{r-M}{(r+M)^3}\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta/2\, o_A - \frac{\mathrm{i}}{\sqrt{2}}\frac{1}{(r+M)^2}\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta/2\,\iota_A\,,$$

$$\lambda_A^1 + \mathrm{i}\lambda_A^2 = \frac{r-M}{(r+M)^3}\mathrm{e}^{-\mathrm{i}\psi}\mathrm{e}^{\mathrm{i}\phi/2}\sin\theta/2\, o_A\,,$$

$$\lambda_A^1 - \mathrm{i}\lambda_A^2 = \frac{-\sqrt{2}}{(r+M)^2}\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{\mathrm{i}\phi/2}\cos\theta/2\,\iota_A\,, \tag{14}$$

$$\mu_A^3 = -\frac{\mathrm{i}}{2}\frac{r-M}{(r+M)^3}\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta/2\, o_A - \frac{\mathrm{i}}{\sqrt{2}}\frac{1}{(r+M)^2}\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta/2\,\iota_A\,,$$

$$\mu_A^1 + \mathrm{i}\mu_A^2 = \frac{r-M}{(r+M)^3}\mathrm{e}^{-\mathrm{i}\phi/2}\cos\theta/2\, o_A\,,$$

$$\mu_A^1 - \mathrm{i}\mu_A^2 = \frac{\sqrt{2}}{(r+M)^2}\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{-\mathrm{i}\phi/2}\sin\theta/2\,\iota_A\,, \tag{15}$$

$$\nu_A^3 = -\frac{2\sqrt{2}\mathrm{i}M}{(r-M)^{1/2}}\frac{\mathrm{e}^{-r/4M}}{(r+M)^2}\mathrm{e}^{\mathrm{i}\chi/2}\,\iota_A\,,$$

$$\nu_A^1 + \mathrm{i}\nu_A^2 = \frac{(r-M)^{1/2}}{(r+M)^2}\left\{1+\frac{4M}{r+M}\right\}\mathrm{e}^{-r/4M}\mathrm{e}^{-\mathrm{i}\chi/2}o_A\,,$$

$$\nu_A^1 - \mathrm{i}\nu_A^2 = 0\,, \tag{16}$$

$$\kappa_A^3 = -\sqrt{2}\mathrm{i}M\frac{(r-M)^{1/2}}{(r+M)^3}\mathrm{e}^{-r/4M}\mathrm{e}^{-\mathrm{i}\chi/2}\,o_A\,,$$

$$\kappa_A^1 + \mathrm{i}\kappa_A^2 = 0\,,$$

$$\kappa_A^1 - \mathrm{i}\kappa_A^2 = \frac{1}{(r-M)^{1/2}}\frac{1}{(r+M)} \times \mathrm{e}^{-r/4M}\left(1+\frac{4M}{r+M}\right)\mathrm{e}^{\mathrm{i}\chi/2}\,\iota_A\,. \tag{17}$$

The basis (o_A, ι_A) is not regular at $r = M$, however we can define a regular basis by

$$o^A_{\mathrm{reg}} = (\rho/4M)\mathrm{e}^{\mathrm{i}\psi/2}\mathrm{e}^{\mathrm{i}\phi/2}\,o^A\,,$$

$$\iota^A_{\mathrm{reg}} = (\rho/4M)^{-1}\mathrm{e}^{-\mathrm{i}\psi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\,\iota^A\,,$$

and it can be seen that the solutions are regular at $r = M$ with respect to this new basis.

We now solve eq. (4) to obtain the constant covariant primed spinors

$$\bar{\alpha}^{A'} = \sin\theta/2\,\mathrm{e}^{\mathrm{i}\phi/2}\,\bar{o}^{A'} + \sqrt{2}\cos\theta/2\,\mathrm{e}^{\mathrm{i}\phi/2}\,\bar{\iota}^{A'}\,,$$

$$\bar{\beta}^{A'} = \cos\theta/2\,\mathrm{e}^{-\mathrm{i}\phi/2}\,\bar{o}^{A'} - \sqrt{2}\sin\theta/2\,\mathrm{e}^{\mathrm{i}\phi/2}\bar{\iota}^{A'}\,. \tag{18}$$

Then taking the product of these with the solutions (14), (15), (16) and (17) will give us the eight zero-mode perturbations.

The three modes corresponding to isospin rotations are found to be $\lambda^i_A\bar{\alpha}_{A'}$, $\mu^i_A\bar{\beta}_{A'}$ and $\lambda^i_A\bar{\beta}_{A'} - \mu^i_A\bar{\alpha}_{A'}$. An interesting mode is $\beta^i_{AA'} = \lambda^i_A\bar{\beta}_{A'} + \mu^i_A\bar{\alpha}_{A'}$ which can be written as

$$B^3 = -2M\frac{r-M}{(r+M)^3}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)\,,$$

$$B^1 = \frac{r-M}{(r+M)^2}(\sin\theta\sin\psi\,\mathrm{d}\phi + \cos\psi\,\mathrm{d}\theta)\,,$$

$$B^2 = \frac{r-M}{(r+M)^2}(\sin\theta\cos\psi\,\mathrm{d}\phi - \sin\psi\,\mathrm{d}\theta)\,. \tag{19}$$

These can be found in another way as the zero mode corresponding to a one-parameter family of solutions to the self-dual Yang–Mills equations found by Pope [5].

The family is given by

$$A^3 = \frac{r-M}{r+M}\left\{1+\frac{2M}{r+a}\right\}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)\,,$$

$$A^1 = -\left(\frac{r-M}{r+a}\right)(\sin\theta\sin\psi\,\mathrm{d}\phi + \cos\psi\,\mathrm{d}\theta)\,,$$

$$A^2 = -\left(\frac{r-M}{r+a}\right)(\sin\theta\cos\psi\,\mathrm{d}\phi - \sin\psi\,\mathrm{d}\theta)\,, \qquad (20)$$

where a varies from M to $+\infty$.

$a = M$ corresponds to the instanton solution above (10) and the limit $a \to \infty$ corresponds to a Dyon solution.

Thus the result found in the first part of the paper is illustrated by the calculation in the second half.

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