Key Exchange

CS 601.642/442 Modern Cryptography

Fall 2019

Groups

- A group G is defined by a set of elements and an operation which maps two elements in the set to a third element
- $(G, \bullet)$ is a group if it satisfies the following conditions:
  - Closure: For all $a, b \in G$, we have $a \bullet b \in G$
  - Associativity: For all $a, b, c \in G$, we have $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
  - Identity: There exists an element e such that for all $a \in G$, we have $e \bullet a = a$
  - Inverse: For every $a \in G$, there exists $b \in G$ such that $a \bullet b = e$
- Think: Is $a \bullet b$ always equal to $b \bullet a$?
  - Read: Abelian Groups
- Example: $(\mathbb{Z}, +)$

Cyclic Groups

- A group $(G, \cdot)$ is a cyclic group if it is generated by a single element
- That is: $G = \{1 = e = g^0, g^1, \ldots, g^{n-1}\}$, where $|G| = n$
- Written as: $G = \langle g \rangle$
- Order of G: n

Discrete Logarithm Problem

- Let $(G, \cdot)$ be a cyclic group of order p with generator g, where p is an n-bit "safe prime" number (i.e., $p = 2q + 1$ for some large prime q).
- Given $(g, b = g^a)$, where $a \xleftarrow{\$} \{0, \ldots, p-1\}$, it is hard to predict a

Discrete Logarithm Problem: Definition

Definition (Discrete Logarithm Problem)

Let $(G, \cdot)$ be a cyclic group of order p (where p is a safe prime) with generator g, then for every non-uniform PPT adversary $\mathcal{A}$, there exists a negligible function ε such that

$$\Pr[a \xleftarrow{\$} \{0, \ldots, p-1\}, a' \leftarrow \mathcal{A}(G, p, g, g^a) : a = a'] \leqslant \varepsilon$$

Computational Diffie-Hellman Assumption

- Let G be a cyclic group $(G, \cdot)$ of order p with generator g, where p is an n-bit safe prime number.
- Give (g, g^a, g^b) to the adversary
- Hard to find g^{ab}

Computational Diffie-Hellman Assumption: Definition

Definition (Computational Diffie-Hellman Assumption)

Let $(G, \cdot)$ be a cyclic group of order p (where p is a safe prime) with generator g, then for every non-uniform PPT adversary $\mathcal{A}$, there exists a negligible function ε such that

$$\Pr[a, b \xleftarrow{\$} \{0, \ldots, p-1\}, y \leftarrow \mathcal{A}(G, p, g, g^a, g^b) : g^{ab} = y] \leqslant \varepsilon$$

Decisional Diffie-Hellman Assumption

- Let $(G, \cdot)$ be a cyclic group of order p with generator g, where p is an n-bit safe prime number.
- Pick $b \xleftarrow{\$} \{0, 1\}$
- If $b = 0$, send (g, g^a, g^b, g^{ab}), where $a, b \xleftarrow{\$} \{0, \ldots, p-1\}$
- If $b = 1$, send (g, g^a, g^b, g^r), where $a, b, r \xleftarrow{\$} \{0, \ldots, p-1\}$
- Adversary has to guess b
- Effectively: $(g, g^a, g^b, g^{ab}) \approx (g, g^a, g^b, g^r)$, for $a, b, r \xleftarrow{\$} \{0, \ldots, p-1\}$ and any g

Decisional Diffie-Hellman Assumption: Definition

Definition (Decisional Diffie-Hellman Assumption)

Let $(G, \cdot)$ be a cyclic group of order p (where p is a safe prime) with generator g, then the following two distributions are computationally indistinguishable:

- $\{a, b \xleftarrow{\$} \{0, \ldots, p-1\} : (G, p, g, g^a, g^b, g^{ab})\}$
- $\{a, b, r \xleftarrow{\$} \{0, \ldots, p-1\} : (G, p, g, g^a, g^b, g^r)\}$

Relationship

$$\text{DDH} \implies \text{CDH} \implies \text{DL}$$

Key Agreement

- Alice and Bob want to share a key.
- They want to establish a shared by by sending each other messages over a channel.
- However, there is an adversary (Eavesdropper) that is eavesdropping on this channel and sees the messages that are sent over it.
- How to securely establish a shared key while keeping it hidden from the eavesdropper?

Key Agreement: Definition

- Alice picks a local randomness r_A
- Bob picks a local randomness r_B
- Alice and Bob engage in a protocol and generate the transcript τ
- Alice's view $V_A = (r_A, \tau)$ and Bob's view $V_B = (r_B, \tau)$
- Eavesdropper's view $V_E = \tau$
- Alice outputs k_A as a function of V_A and Bob outputs k_B as a function of V_B
- Correctness: $\Pr_{r_A, r_B}[k_A = k_B] \approx 1$
- Security: $(k_A, V_E) \equiv (k_B, V_E) \approx (r, \tau)$

Key Agreement: Construction (Diffie-Hellman)

- Let $(G, \cdot)$ be a cyclic group of order p (where p is a safe prime) with generator g.
- Alice picks $a \xleftarrow{\$} \{0, \ldots, p-1\}$ and sends g^a to Bob
- Bob picks $b \xleftarrow{\$} \{0, \ldots, p-1\}$ and sends g^b to Alice
- Alice outputs $(g^b)^a$ and Bob outputs $(g^a)^b$
- Adversary sees: (g^a, g^b)
- Correctness?
- Security? Use DDH to say that g^{ab} is hidden from adversary's view
- Think: Is this scheme still secure if the adversary is allowed to modify the messages?