Alice wants to prove an NP statement to Bob without revealing her private witness.

However, Alice only has the resource to send a single message to Bob. Therefore, they cannot run an interactive zero-knowledge proof.

To make matters worse, 1-message zero-knowledge is only possible for languages in BPP! (Think: Why?)

Fortunately, they both have access to a common random string that was (honestly) generated by someone they both trust.

Can Alice prove statements non-interactively to Bob using the common random string?
Non-Interactive Proofs

Syntax. A non-interactive proof system for a language $L$ with witness relation $R$ is a tuple of algorithms $(K, P, V)$ such that:

- **Setup:** $\sigma \leftarrow K(1^n)$ outputs a common random string
- **Prove:** $\pi \leftarrow P(\sigma, x, w)$ takes as input a common random string $\sigma$, a statement $x \in L$ and a witness $w$ and outputs a proof $\pi$
- **Verify:** $V(\sigma, x, \pi)$ outputs 1 if it accepts the proof and 0 otherwise

A non-interactive proof system must satisfy completeness and soundness properties.
Non-Interactive Proofs (contd.)

Completeness: \( \forall x \in L, \forall w \in R(x) \):

\[
\Pr \left[ \sigma \leftarrow K(1^n); \pi \leftarrow P(\sigma, x, w) : V(\sigma, x, \pi) = 1 \right] = 1
\]

Non-Adaptive Soundness: There exists a negligible function \( \nu(\cdot) \) s.t.
\( \forall x \notin L \):

\[
\Pr \left[ \sigma \leftarrow K(1^n); \exists \pi \text{ s.t. } V(\sigma, x, \pi) = 1 \right] \leq \nu(n)
\]

Adaptive Soundness: There exists a negligible function \( \nu(\cdot) \) s.t.:

\[
\Pr \left[ \sigma \leftarrow K(1^n); \exists (x, \pi) \text{ s.t. } x \notin L \land V(\sigma, x, \pi) = 1 \right] \leq \nu(n)
\]

Note: In non-adaptive soundness, the adversary chooses \( x \) before seeing the common random string whereas in adaptive soundness, it can choose \( x \) depending upon the common random string.
Non-Interactive Zero Knowledge (NIZK)

Definition (Non-Adaptive NIZK)

A non-interactive proof system \((K, P, V)\) for a language \(L\) with witness relation \(R\) is *non-adaptive zero-knowledge* if there exists a PPT simulator \(S\) s.t. for every \(x \in L\), \(w \in R(x)\), the output distributions of the following two experiments are computationally indistinguishable:

<table>
<thead>
<tr>
<th>REAL((1^n, x, w))</th>
<th>IDEAL((1^n, x))</th>
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</thead>
<tbody>
<tr>
<td>(\sigma \leftarrow K(1^n))</td>
<td>((\sigma, \pi) \leftarrow S(1^n, x))</td>
</tr>
<tr>
<td>(\pi \leftarrow P(\sigma, x, w))</td>
<td></td>
</tr>
<tr>
<td>Output ((\sigma, \pi))</td>
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**Note:** The simulator generates both the common random string and the simulated proof given the statement \(x\) is input. In particular, the simulated common random string can depend on \(x\) and can therefore only be used for a single proof.
Non-Interactive Zero Knowledge (contd.)

**Definition (Adaptive NIZK)**

A non-interactive proof system \((K, P, V)\) for a language \(L\) with witness relation \(R\) is *adaptive zero-knowledge* if there exists a PPT simulator \(S = (S_0, S_1)\) s.t. for every \(x \in L\), \(w \in R(x)\), the output distributions of the following two experiments are computationally indistinguishable:

- **REAL**\((1^n, x, w)\)
  - \(\sigma \leftarrow K(1^n)\)
  - \(\pi \leftarrow P(\sigma, x, w)\)
  - Output \((\sigma, \pi)\)

- **IDEAL**\((1^n, x)\)
  - \((\sigma, \tau) \leftarrow S_0(1^n)\)
  - \(\pi \leftarrow S_1(\sigma, \tau, x)\)
  - Output \((\sigma, \pi)\)

**Note 1:** Here, \(\tau\) is a “trapdoor” for the simulated common random string \(\sigma\) that is used by \(S_1\) to generate an accepting proof for \(x\) without knowing the witness.

**Note 2:** This definition captures *reusable* common random strings
Remarks on NIZK Definition

- In NIZK, the simulator is given “extra power” to choose the common random string, along with possibly a trapdoor to enable simulation without a witness.
- In interactive ZK, the extra power to the simulator was the ability to “reset” the verifier.
- Indeed, a simulator must always have some extra power over the normal prover, otherwise, the definition would be impossible to realize for languages outside \( \text{BPP} \).
- In NIZKs, the extra power is ok since we require indistinguishability of the “joint distribution” over the common random string and the proof.
Lemma

There exists an efficient transformation from any non-interactive proof system \((K, P, V)\) with non-adaptive soundness into a non-interactive proof system \((K', P', V')\) with adaptive soundness.

Proof Strategy: Let \(\ell(n)\) be the length of the statements.

- Repeat \((K, P, V)\) polynomially many times (with fresh randomness) so that soundness error decreases to \(2^{-2\ell(n)}\).
- Non-adaptive soundness means that a randomly sampled \(\sigma\) is “bad” for a statement \(x\) with probability \(2^{-2\ell(n)}\).
- By Union Bound, \(\sigma\) is “bad” for all statements with probability \(2^{-\ell(n)}\). Therefore, we have adaptive soundness.
I. Non-adaptive Zero Knowledge: We first construct NIZKs for NP with non-adaptive zero-knowledge property using the following two steps:

   **Step 1.** Construct a NIZK proof system for NP in the hidden-bit model. This step is unconditional.

   **Step 2.** Using trapdoor permutations, transform any NIZK proof system for language in the hidden-bit model to a non-adaptive NIZK proof system in the common random string model.

II. Adaptive Zero Knowledge: Next, we transform non-adaptive NIZKs for NP into adaptive NIZKs for NP. This step only requires one-way functions, which are implied by trapdoor permutations.

Putting all the steps together, we obtain adaptive NIZKs for NP based on trapdoor permutations.
Roadmap

- **Today:** Defining NIZKs in hidden-bit model, and transformation from NIZKs in hidden-bit model to NIZKs in common random string model

- **Next time:** NIZKs for NP in the hidden-bit model

- **Homework:** Non-adaptive NIZKs to Adaptive NIZKs
NIZK in Hidden-Bit Model

**Syntax.** A non-interactive proof system for a language $L$ with witness relation $R$ in the hidden-bit model is a tuple of algorithms $(K_{HB}, P_{HB}, V_{HB})$ such that:

- **Setup:** $r \leftarrow K_{HB}(1^n)$ outputs the hidden random string
- **Prove:** $(I, \pi) \leftarrow P_{HB}(r, x, w)$ generates the indices $I \subseteq [|r|]$ of $r$ to reveal, along with a proof $\pi$
- **Verify:** $V_{HB}(I, \{r_i\}_{i \in I}, \pi)$ outputs 1 if it accepts the proof and 0 otherwise

Such a proof system must satisfy completeness and soundness (similar to as defined earlier)
A non-interactive proof system \((K_{\text{HB}}, P_{\text{HB}}, V_{\text{HB}})\) for a language \(L\) with witness relation \(R\) in the hidden-bit model is (non-adaptive) zero-knowledge if there exists a PPT simulator \(S_{\text{HB}}\) s.t. for every \(x \in L\), \(w \in R(x)\), the output distributions of the following two experiments are computationally indistinguishable:

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**Intuition:** How to transform a “public” random string into a “hidden” random string

- Suppose the prover samples a trapdoor permutation \((f, f^{-1})\) with hardcore predicate \(h\)
- Given a common random string \(\sigma = \sigma_1, \ldots, \sigma_n\), the prover can compute \(r = r_1, \ldots, r_n\) where:
  \[
  r_i = h(f^{-1}(\sigma_i))
  \]
- If \(f\) is a permutation and \(h\) is a hard-core predicate, then \(r\) is guaranteed to be random
- Now \(r\) can be treated as the hidden random string: \(V\) can only see the parts of it that the prover wishes to reveal
Construction

Let $\mathcal{F} = \{f, f^{-1}\}$ be a family of $2^n$ trapdoor permutations with hardcore predicate $h$. Let $(K_{\text{HB}}, P_{\text{HB}}, V_{\text{HB}})$ be a NIZK proof system for $L$ in the hidden-bit model with soundness error $2^{-2n}$.

**Construction of $(K, P, V)$:**

- **$K(1^n)$**: Output a random string $\sigma = \sigma_1, \ldots, \sigma_n$ s.t. $\forall i, |\sigma_i| = n$.
- **$P(\sigma, x, w)$**: Execute the following steps:
  - Sample $(f, f^{-1}) \leftarrow \mathcal{F}(1^n)$
  - Compute $\alpha_i = f^{-1}(\sigma_i)$ for $i \in [n]$
  - Compute $r_i = h(\alpha_i)$ for $i \in [n]$
  - Compute $(I, \phi) \leftarrow P_{\text{HB}}(r, x, w)$
  - Output $\pi = (f, I, \{\alpha_i\}_{i \in I}, \Phi)$

- **$V(\sigma, x, \pi)$**: Parse $\pi = (f, I, \{\alpha_i\}_{i \in I}, \Phi)$ and:
  - Check $f \in \mathcal{F}$ and $f(\alpha_i) = \sigma_i$ for every $i \in I$
  - Compute $r_i = h(\alpha_i)$ for $i \in I$
  - Output $V_{\text{HB}}(I, \{r_i\}_{i \in I}, x, \Phi)$
$(K, P, V)$ is a Non-Interactive Proof

- **Completeness:** $\alpha$ is uniformly distributed since $f^{-1}$ is a permutation and $\sigma$ is random. Further, since $h$ is a hard-core predicate, $r$ is also uniformly distributed. Completeness follows from the completeness of $(K_{HB}, P_{HB}, V_{HB})$

- **Soundness:** For any $f = f_0$, $r$ is uniformly random, so from (non-adaptive) soundness of $(K_{HB}, P_{HB}, V_{HB})$, we have:

  $$\Pr_{\sigma}[P^* \text{ can cheat using } f_0] \leq 2^{-2n}$$

  Since there are only $2^n$ possible choices of $f$ (verifier checks that $f \in \mathcal{F}$), by union bound, it follows:

  $$\Pr_{\sigma}[P^* \text{ can cheat }] \leq 2^{-n}$$
Let $S_{HB}$ be the simulator for $(K_{HB}, P_{HB}, V_{HB})$

**Simulator $S(1^n, x)$:**

1. $(I, \{r_i\}_{i \in I}, \Phi) \leftarrow S_{HB}(1^n, x)$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in I$
4. $\sigma_i = f(\alpha_i)$ for every $i \in I$
5. $\sigma_i \leftarrow \{0, 1\}^n$ for every $i \notin I$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

**Note:** $h^{-1}(r_i)$ denotes sampling from the pre-image of $r_i$, which can be done efficiently by simply trying random $\alpha_i$'s until $h(\alpha_i) = r_i$
Proof of Zero Knowledge: Hybrids

**Hybrid** $H_0(1^n, x, w) := \text{REAL}(1^n, x, w)$:

1. $\sigma \leftarrow K(1^n)$ where $\sigma = \sigma_1, \ldots, \sigma_n$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow f^{-1}(\sigma_i)$ for every $i \in [n]$
4. $r_i = h(\alpha_i)$ for every $i \in [n]$
5. $(I, \Phi) \leftarrow P_{\text{HB}}(r, x, w)$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$
Proof of Zero Knowledge: Hybrids (contd.)

Hybrid $H_1(1^n, x, w)$:

1. $\alpha _i \leftarrow \{0, 1\}^n$ for every $i \in [n]$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\sigma _i \leftarrow f(\alpha _i)$ for every $i \in [n]$
4. $r_i = h(\alpha _i)$ for every $i \in [n]$
5. $(I, \Phi) \leftarrow P_{\text{HB}}(r, x, w)$
6. Output $(\sigma, f, I, \{\alpha _i\}_{i \in I}, \Phi)$

$H_0 \approx H_1$: In $H_1$, we sample $\alpha _i$ at random and then compute $\sigma _i$ (instead of sampling $\sigma _i$ and then computing $\alpha _i$ as in $H_0$). This induces an identical distribution since $f$ is a permutation.
Hybrid $H_2(1^n, x, w)$:

1. $r_i \leftarrow \{0, 1\}$ for every $i \in [n]$ 
2. $(f, f^{-1}) \leftarrow \mathcal{F}$ 
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in [n]$ 
4. $\sigma_i = f(\alpha_i)$ for every $i \in [n]$ 
5. $(I, \Phi) \leftarrow P_{HB}(r, x, w)$ 
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_1 \approx H_2$: In $H_2$, we again change the sampling order: first sample $r = r_1, \ldots, r_n$ at random and then sample $\alpha_i$ from the pre-image of $r_i$ (as described earlier). This distribution is identical to $H_1$. 
Hybrid $H_3(1^n, x, w)$:

1. $r_i \leftarrow \{0, 1\}$ for every $i \in [n]$
2. $(f, f^{-1}) \leftarrow \mathcal{F}$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in [n]$
4. $(I, \Phi) \leftarrow \text{P HB}(r, x, w)$
5. $\sigma_i = f(\alpha_i)$ for every $i \in I$
6. $\sigma_i \leftarrow \{0, 1\}^n$ for every $i \notin I$
7. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_2 \approx_c H_3$: In $H_3$, we output random $\sigma_i$ for $i \in I$. From security of hard-core predicate $h$, it follows that:

$$\{f(h^{-1}(r_i))\} \approx_c U_n$$

Indistinguishability of $H_2$ and $H_3$ follows using the above equation.
Proof of Zero Knowledge: Hybrids (contd.)

Hybrid $H_4(1^n, x) := \text{IDEAL}(1^n, x)$:

1. $(I, \{r_i\}_{i \in I}, \Phi) \leftarrow S_{HB}(1^n, x)$
2. $(f, f^{-1}) \leftarrow F$
3. $\alpha_i \leftarrow h^{-1}(r_i)$ for every $i \in I$
4. $\sigma_i = f(\alpha_i)$ for every $i \in I$
5. $\sigma_i \leftarrow \{0, 1\}^n$ for every $i \notin I$
6. Output $(\sigma, f, I, \{\alpha_i\}_{i \in I}, \Phi)$

$H_3 \approx_c H_4$: In $H_4$, we swap $P_{HB}$ with $S_{HB}$. Indistinguishability follows from the zero-knowledge property of $(K_{HB}, P_{HB}, V_{HB})$