High-Dimensional Multi-Model Estimation – Its Algebra, Statistics, and Sparse Representation

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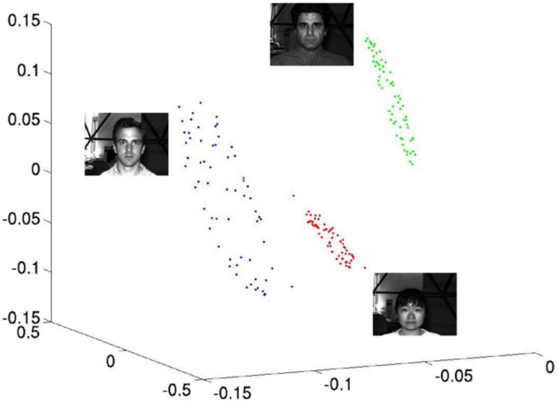
High-Dimensional Data: Images, Videos, etc...

Figure: Dimension of an image: $1000 \times 700 \times 3 > 2\textit{million}$!

HD data are often **multi-model**

Recognition of Multi-Model Data



Face Recognition: *"Where amazing happens!"*

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Figure: Kevin Garnett, Steve Nash, Jason Kidd, Yao Ming.

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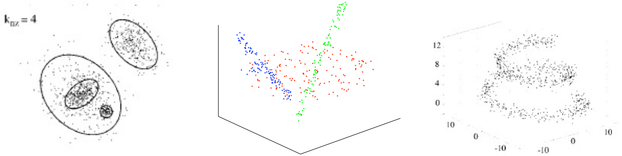
- How to determine a class of models and the number of models?



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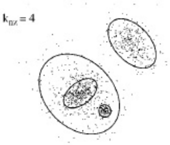
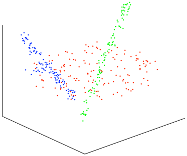
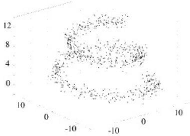
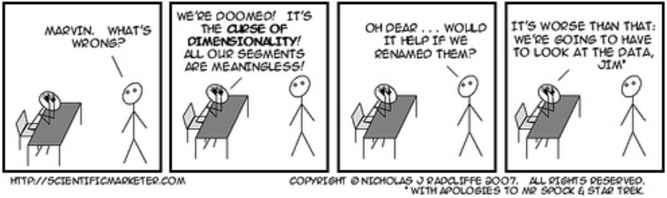






- Curse of dimensionality! [Richard Bellman 1957]



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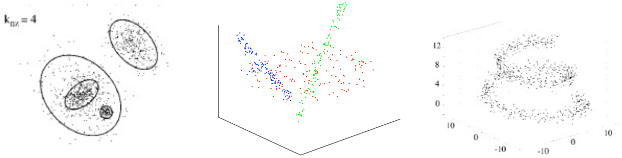

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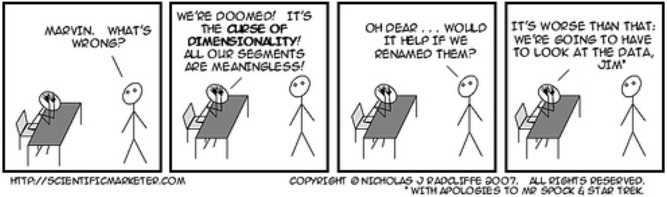
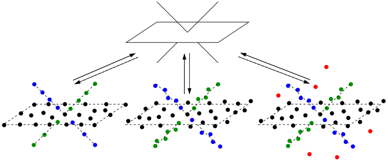


- To make things worse: Robust to high noise and outliers?

Pattern Analysis of Multiple Geometric Models

1. **Unsupervised** segmentation
   Segment samples drawn from $\mathcal{A} = S_1 \cup S_2 \cup \ldots \cup S_K$ in $\mathbb{R}^D$, and estimate model parameters.



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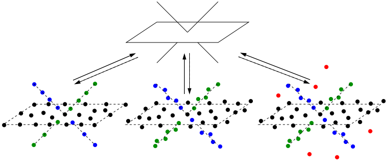
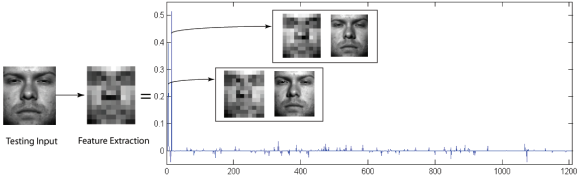


2. **Supervised** recognition
   Assume training examples $\{A_1, \cdots, A_K\}$ for K models. Given a test sample $\mathbf{y}$, determine its membership label$(\mathbf{y}) \in [1, 2, \cdots, K]$.



Affine Motion Segmentation

Assume multiple 3-D objects far away from the camera in a dynamic scene

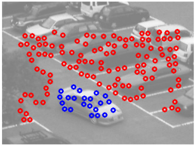
- 3-D features $\mathbf{p}_1, \ldots, \mathbf{p}_N \in \mathbb{R}^3$ are tracked in F image frames.
- Image of $\mathbf{p}_i$ in jth frame:

$$\mathbf{m}_{ij} \doteq \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix}^T = A_j \mathbf{p}_i + \mathbf{b}_j \in \mathbb{R}^2, \quad j = 1, \ldots, F.$$

- Stack images of $\mathbf{p}_i$ in all F frames

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{m}_{i1} \\ \vdots \\ \mathbf{m}_{iF} \end{bmatrix} = \begin{bmatrix} A_1 & \mathbf{b}_1 \\ \vdots & \\ A_F & \mathbf{b}_F \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} \in \mathbb{R}^{2F}.$$

parking-lot movie



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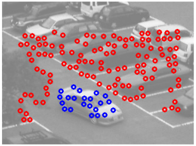


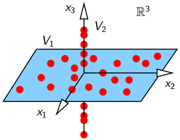
Challenge: Affine Motion Segmentation

Each motion satisfies a 4-D subspace model. Therefore motion segmentation becomes **subspace segmentation problem**.

Generalized Principal Component Analysis (GPCA)

1. For a single subspace

- $V_1^\perp : (x_3 = 0)$
- $V_2^\perp : (x_1 = 0)\&(x_2 = 0)$



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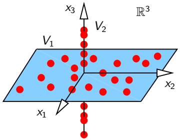

2. For $\mathcal{A} = V_1 \cup V_2$

$$\forall \mathbf{z} = (x_1, x_2, x_3)^T, \quad \mathbf{z} \in V_1 \cup V_2 \Leftrightarrow \{x_3 = 0\} | \{(x_1 = 0) \& (x_2 = 0)\}$$

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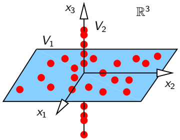

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3. By De Morgan's law

$$\{x_3 = 0\} | \{(x_1 = 0)\&(x_2 = 0)\} \Leftrightarrow (x_1 x_3 = 0)\&(x_2 x_3 = 0) \Leftrightarrow \begin{cases} x_1 x_3 = 0 \\ x_2 x_3 = 0 \end{cases}$$

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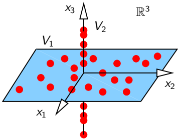

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4. **Vanishing polynomials**: $p_1 = x_1 x_3, p_2 = x_2 x_3$

Equivalence Relation

- The **equivalence** between K subspaces and Kth-degree vanishing polynomials
  1. Given $p_1 = x_1 x_3, p_2 = x_2 x_3$, $V_1 \cup V_2$ uniquely determined.
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- Kth-degree vanishing polynomials $I_K(\mathcal{A})$ as a **global signature**
- $I_K(\mathcal{A})$ is a **polynomial subspace**.

Subspace Properties

If $p_1(\mathbf{x}) = 0$ and $p_2(\mathbf{x}) = 0$

1. Closed under **addition**: $(p_1 + p_2)(\mathbf{x}) = 0 \Rightarrow (p_1 + p_2) \in I_K(\mathcal{A})$.
2. Closed under **scalar multiplication**: $\forall a \in \mathbb{R}$, $ap_1(\mathbf{x}) = 0$ and $ap_2(\mathbf{x}) = 0 \Rightarrow ap_1, ap_2 \in I_K(\mathcal{A})$.

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- $I_K(\mathcal{A})$ is determined by a linearly-independent polynomial basis.

Estimation of Vanishing Polynomials

1. **Veronese embedding**: Given N samples $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^3$,

$$L_2 \doteq [\nu_2(\mathbf{x}_1), \ldots, \nu_2(\mathbf{x}_N)] \in \mathbb{R}^{M_2^{[3]} \times N}$$

$$= \begin{bmatrix} \cdots & (x_1)^2 & \cdots \\ \cdots & (x_1 x_2) & \cdots \\ \cdots & (x_1 x_3) & \cdots \\ \cdots & (x_2)^2 & \cdots \\ \cdots & (x_2 x_3) & \cdots \\ \cdots & (x_3)^2 & \cdots \end{bmatrix}$$

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2. The null space of L_2 is $\begin{array}{l} \mathbf{c}_1 = [0,0,1,0,0,0] \\ \mathbf{c}_2 = [0,0,0,0,1,0] \end{array} \Rightarrow \begin{array}{l} p_1 = \mathbf{c}_1 \nu_2(\mathbf{x}) = x_1x_3 \\ p_2 = \mathbf{c}_2 \nu_2(\mathbf{x}) = x_2x_3 \end{array}$

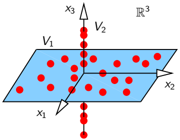

Figure: 2*nd*-degree vanishing polynomials: $p_1 = x_1x_3$, $p_2 = x_2x_3$.

Calculate Subspace Basis Vectors using Polynomial Derivatives

1. $V_1^{\perp}, \cdots, V_K^{\perp}$ recovered by the *derivatives*

$$\nabla_{\mathbf{x}} P = [\, \nabla_{\mathbf{x}} p_1 \;\; \nabla_{\mathbf{x}} p_2 \,] = \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \\ x_1 & x_2 \end{bmatrix}.$$

2. Pick $\mathbf{z} = [1, 1, 0]^T \in V_1$, then $\nabla_{\mathbf{x}} P(\mathbf{z}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

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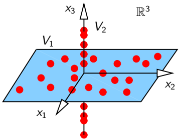

Figure: $P(\mathbf{x}) \doteq [p_1(\mathbf{x}) \;\; p_2(\mathbf{x})] = [x_1 x_3, x_2 x_3]$.

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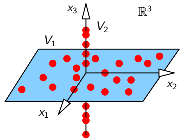

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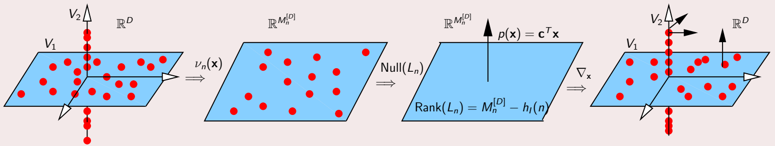
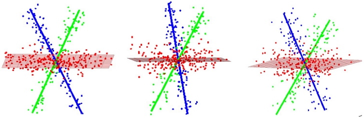
Diagram of GPCA

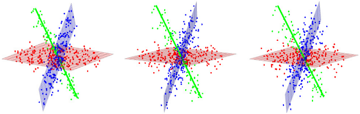


Robust GPCA



(a) .08 (b) .12 (c) .16

Figure: (2, 1, 1) with various noise-to-signal ratios



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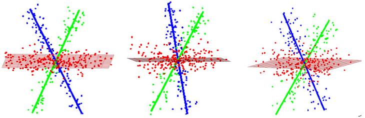

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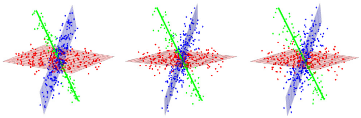

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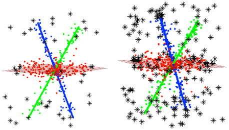

(a) 12% (b) 32%

Figure: One plane and two lines with various outlier percentages

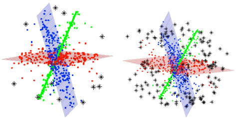

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Figure: Two planes and one line with various outlier percentages

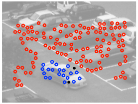
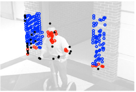
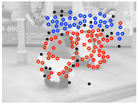
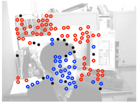
Outlier Elimination

Figure: Elimination of outliers.

Experiment: Affine Motion Segmentation

Sequences:

RGPCA:

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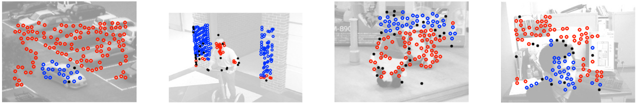
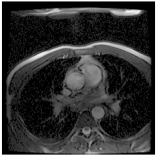
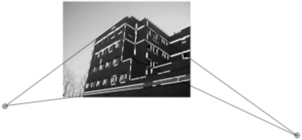


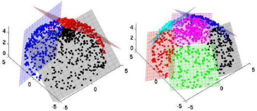
- Other applications



(d) Image/Video Segmentation



(e) Vanishing Point Detection



(f) Manifold Fitting

Reference:

SIAM Review: *Estimation of subspace arrangements with applications in modeling and segmenting mixed data*, 2008.

Summary: GPCA

Advantages:

- Closed-form algebraic solution, not iterative.
- Segmentation of subspaces with mixed dimensions.
- Robust to noise and outliers.

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- **User provides correct subspace number and dimensions. (How to select a good mixture model?)**

Mixture Perspective Motions

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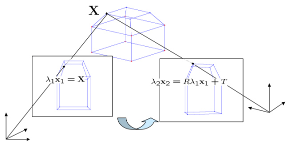
Given two image correspondences $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$

- Epipolar



$$\mathbf{x}_2^T \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \mathbf{x}_1 = 0$$

Mixture Perspective Motions

Given two image correspondences $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$

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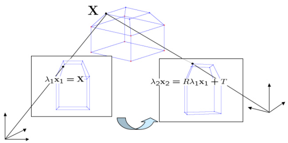

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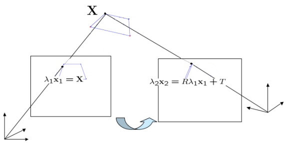

$$\mathbf{x}_2 \times \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}_1 = 0$$

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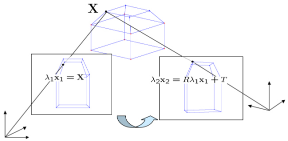

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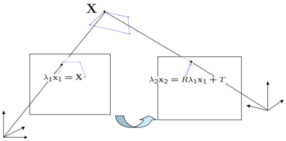

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Segmentation of mixture perspective motions

Each perspective constraint is linear w.r.t. $(\mathbf{x}_1, \mathbf{x}_2)$, but in different form!

Quadratic Manifolds in Joint Image Space

Joint image space: Stack $\mathbf{x}_1 = (x_1, y_1, 1)^T$ and $\mathbf{x}_2 = (x_2, y_2, 1)^T$

$$\mathbf{y} = (x_1, y_1, x_2, y_2, 1)^T \in \mathbb{R}^5$$

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- Quadratic fundamental manifold (QFM)

$$\mathbf{y}^T A \mathbf{y} \doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & f_{11} & f_{21} & f_{31} \\ 0 & 0 & f_{12} & f_{22} & f_{32} \\ f_{11} & f_{12} & 0 & 0 & f_{13} \\ f_{21} & f_{22} & 0 & 0 & f_{23} \\ f_{31} & f_{32} & f_{13} & f_{23} & 2f_{33} \end{pmatrix} \mathbf{y} = 0. \quad (1)$$

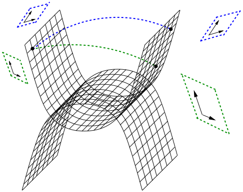
- Quadratic homograpy manifold (QHM)

$$\mathbf{y}^T B_1 \mathbf{y} \doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & 0 & h_{31} & -h_{21} \\ 0 & 0 & 0 & h_{32} & -h_{22} \\ 0 & 0 & 0 & 0 & 0 \\ h_{31} & h_{32} & 0 & 0 & h_{33} \\ -h_{21} & -h_{22} & 0 & h_{33} & -2h_{23} \end{pmatrix} \mathbf{y} = 0,$$
$$\mathbf{y}^T B_2 \mathbf{y} \doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & -h_{31} & 0 & h_{11} \\ 0 & 0 & -h_{32} & 0 & h_{12} \\ -h_{31} & -h_{32} & 0 & 0 & -h_{33} \\ 0 & 0 & 0 & 0 & 0 \\ h_{11} & h_{12} & -h_{33} & 0 & 2h_{13} \end{pmatrix} \mathbf{y} = 0, \quad (2)$$
$$\mathbf{y}^T B_3 \mathbf{y} \doteq \mathbf{y}^T \begin{pmatrix} 0 & 0 & h_{21} & -h_{11} & 0 \\ 0 & 0 & h_{22} & -h_{12} & 0 \\ h_{21} & h_{22} & 0 & 0 & h_{23} \\ -h_{11} & -h_{12} & 0 & 0 & -h_{13} \\ 0 & 0 & h_{23} & -h_{13} & 0 \end{pmatrix} \mathbf{y} = 0.$$

Segmentation of Quadratic Manifolds

- Convert mixture perspective motion as **segmentation of mixture quadratic manifolds** defined by

$$p_j(\mathbf{y}) \doteq \mathbf{y}^T Q_j \mathbf{y} = 0. \quad (3)$$



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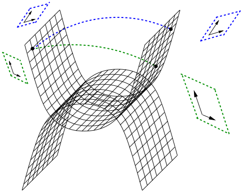

- **Vanishing polynomials** (as global signature): The set of $2K$th degree polynomials $I_{2K}(\mathcal{A})$ **uniquely determines** $\mathcal{A} = S_1 \cup \cdots \cup S_K$.

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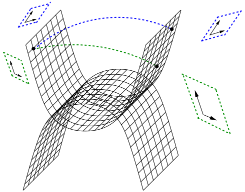

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Robust Algebraic Segmentation

$$Y = \{\mathbf{y}_1, \cdots, \mathbf{y}_n\} \Rightarrow I_{2K}(\mathcal{A}) \Rightarrow \mathcal{A} \Rightarrow \{S_1, \cdots, S_K\}$$

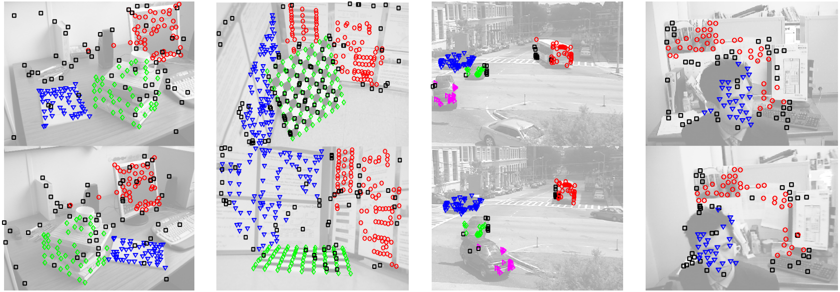
Reference:

IJCV (draft): *Robust Algebraic Segmentation of Mixed Rigid-Body and Planar Motions*, 2008.

boxes	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	9.24%	**0.84%**	1.68%	**0.84%**
VR	36.97%	84.87%	100%	87.39%
carsnbus3	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	45.75%	12.55%	2.83%	**1.62%**
VR	83.81%	90.28%	97.17%	85.83%
deliveryvan	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	23.23%	10.63%	5.91%	**0.39%**
VR	97.64%	96.85%	100%	94.09%
desk	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	9.00%	2.50%	3.00%	**0.50%**
VR	55.50%	93.50%	91.50%	93.50%
lightbulb	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	39.52%	**0.00%**	**0.00%**	**0.00%**
VR	76.19%	82.86%	100%	99.52 %
manycars	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	30.56%	22.22%	**0.00%**	**0.00%**
VR	90.28%	95.83%	100%	88.89%
man-in-office	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	20.56%	34.58%	20.56%	**11.21%**
VR	89.72%	95.33%	84.11%	82.24%
nrbooks3	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	12.38%	9.05%	5.48%	**0.95%**
VR	41.19%	65.48%	94.29%	88.33%
office	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	2.28%	0.33%	10.42%	**0.00%**
VR	89.59%	90.55%	86.97%	93.49%
parking-lot	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	7.86%	5.00%	3.57%	**2.86%**
VR	98.57%	96.43%	100%	97.86%
posters-checkerboard	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	20.58%	1.06%	9.23%	**0.00%**
VR	49.87%	97.36%	70.71%	95.25%
posters-keyboard	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	8.59%	**0.25%**	10.61%	0.51%
VR	56.06%	83.33%	78.03%	88.13%
toys-on-table	MLESAC	MC-RANSAC	RAS	RAS+RANSAC
FPR	38.10%	38.10%	15.08%	**7.94%**
VR	91.27%	92.86%	81.75%	77.78%

Experiment

1. Visualization



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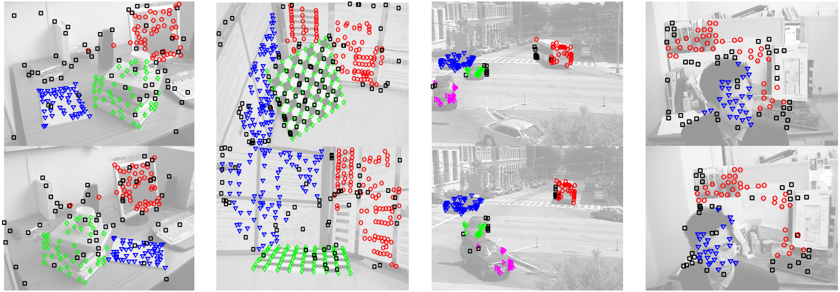
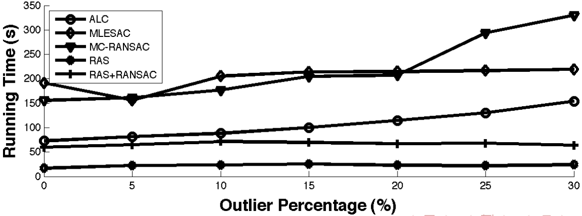


2. Faster than RANSAC!



Summary: Robust Algebraic Segmentation

Advantages:

- **Segmentation of quadratic manifolds with mixed dimensions.**
- Closed-form algebraic solution, not iterative.
- Robust to noise and outliers.

Limitations:

- **User provides correct subspace number and dimensions. (How to select a good mixture model?)**

Lossy Minimum Description Length (LMDL)

1. **Lossy coding length** $L_\epsilon(V, \mathcal{A})$:
   Quantize $V = (v_1, \cdots, v_N) \in \mathbb{R}^{D \times N}$ as **a sequence of binary bits** up to a distortion $\mathbb{E}[\|v_i - \hat{v}_i\|^2] \leq \epsilon^2$.

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   - Model V_i as a (degenerate) Gaussian model
$$\textbf{Bit rate: } R(V_i) = \frac{1}{2}\log_2 \det(I + \frac{D}{\epsilon^2 N_i} V_i V_i^T).$$

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   - **Coding length** for V_i of N_i samples
$$L(V_i) = (N_i + D)R(V_i) + \frac{D}{2}\log_2 \det(1 + \frac{1}{\epsilon^2}\mu_i\mu_i^T) + N_i(-\log_2(N_i/N)).$$

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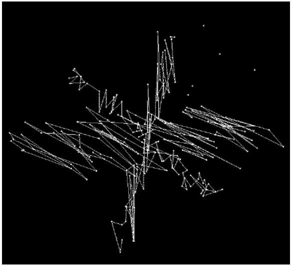
   - **Total coding length**: $L^s(V_1, \cdots, V_K) = \sum_i L(V_i)$.

A Greedy Optimization

1. **Initialize**: Assume N samples as individual groups.
2. Each iteration: Merge two groups that reduces largest coding length.
3. To stop: If any further merging cannot reduces L^s.
4. **Output**: Estimation of K and the grouping.

animation



Simulation

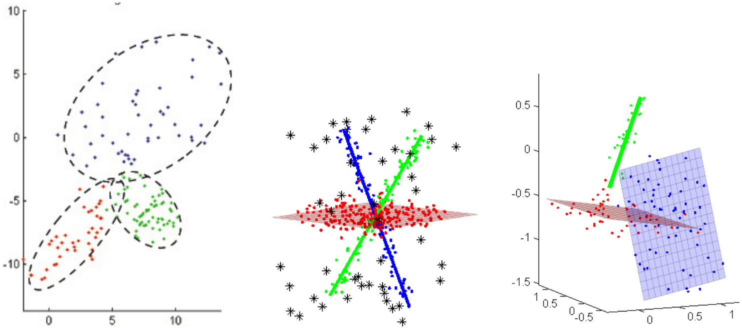
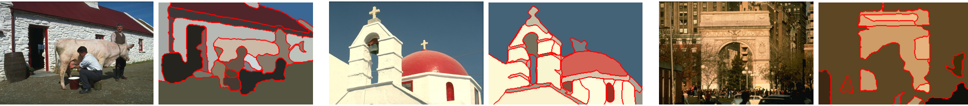


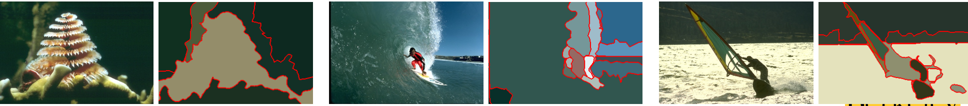
Image Segmentation via Mixture Subspace Models

(g) Nature



(h) Urban

(i) Portraits



(j) Water

Quantitative Comparison

Table: Average performance on the Berkeley image segmentation database.

	PRI	VoI	GCE	BDE
Humans	0.8754	1.1040	0.0797	4.994
$CTM_{\gamma=0.1}$	0.7561	**2.4640**	**0.1767**	**9.4211**
Mean-Shift [Comaniciu 2002]	0.7550	2.477	0.2598	9.7001
N-Cuts [Shi 2000]	0.7229	2.9329	0.2182	9.6038
F-H [Felzenszwalb 2004]	**0.7841**	2.6647	0.1895	9.9497

PRI: Probabilistic Rand Index [Pantofaru 2005].
VoI: Variation of Information [Meila 2005].
GCE: Global Consistency Error [Martin 2001].
BDE: Boundary Displacement Error [Freixenet 2002].

Reference:

Unsupervised Segmentation of Natural Images via Lossy Data Compression, *CVIU*, 2008.

Classification of Mixture Subspaces

- **Notation**
  - Training: For K classes, collect training samples $\{\mathbf{v}_{1,1}, \cdots, \mathbf{v}_{1,n_1}\}, \cdots, \{\mathbf{v}_{K,1}, \cdots, \mathbf{v}_{K,n_K}\} \in \mathbb{R}^D$.
  - Test: Present a new $\mathbf{y} \in \mathbb{R}^D$, solve for label($\mathbf{y}$) $\in [1, 2, \cdots, K]$.

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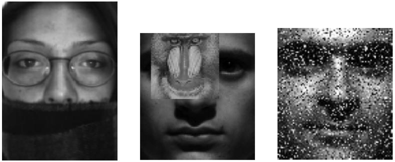
- **Facial disguise & occlusion**



Sparse Representation

Sparsity

A signal is sparse if most of its coefficients are (approximately) zero.

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1. Sparsity in frequency domain

Figure: 2-D DCT transform.

2. Sparsity in spatial domain

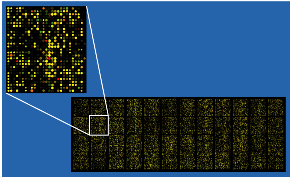
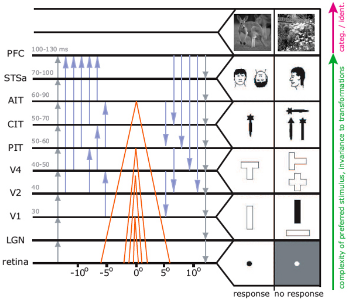


Figure: Gene microarray data.

- **Sparsity in human visual cortex** [Perrett & Oram 1993, Olshausen & Field 1997, Riesenhuber & Poggio 2000]



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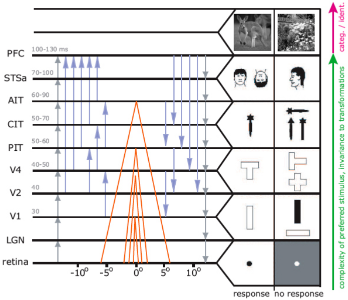
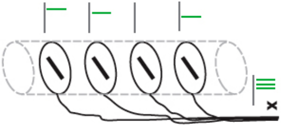
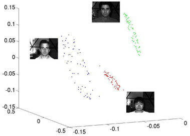


1. **Feed-forward**: No iterative feedback loop.
2. **Redundancy**: Average 80-200 neurons for each feature representation.
3. **Recognition**: Information exchange between stages is not about individual neurons, but rather how many neurons as a group fire together.



Classification of Mixture Subspace Model

1. Face-subspace model: Assume $\mathbf{y}$ belongs to Class i



$$\begin{aligned} \mathbf{y} &= \alpha_{i,1}\mathbf{v}_{i,1} + \alpha_{i,2}\mathbf{v}_{i,2} + \cdots + \alpha_{i,n_1}\mathbf{v}_{i,n_i}, \\ &= A_i\alpha_i, \end{aligned}$$

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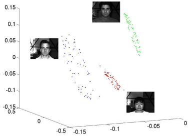

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2. Nevertheless, Class i is the **unknown** variable we need to solve:

Sparse representation $\mathbf{y} = [A_1, A_2, \cdots, A_K] \begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_K\end{bmatrix} = A\mathbf{x}.$

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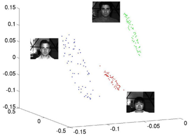

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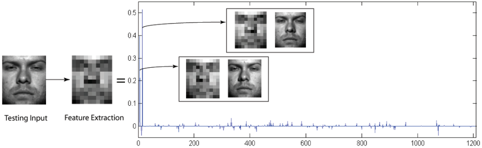
3. $\mathbf{x}_0 = [\,0 \cdots 0 \; \alpha_i^T \; 0 \cdots 0\,]^T \in \mathbb{R}^n.$



Sparse representation x_0 encodes membership!

ℓ^1-Minimization

1. **Ideal solution:** ℓ^0-Minimization

$$(P_0) \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ s.t. } \mathbf{y} = A\mathbf{x}.$$

$\|\cdot\|_0$ simply counts the number of nonzero terms.
However, generally ℓ^0-minimization is *NP-hard*.

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   $\|\cdot\|_0$ simply counts the number of nonzero terms.
   However, generally ℓ^0-minimization is *NP-hard*.

2. **Compressive Sensing**: Under mild condition, ℓ^0-minimization is equivalent to

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ℓ^1-Minimization

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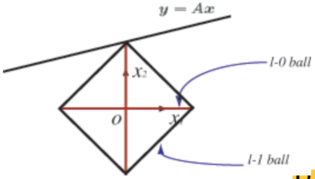
3. ℓ^1-Ball

ℓ^0/ℓ^1 Equivalence

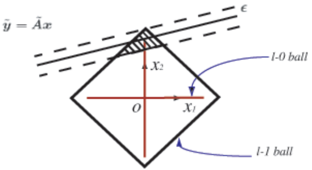
- ℓ^1-Minimization is convex.
- Solution equal to ℓ^0-minimization.



Stability of ℓ^1-Minimization

- ℓ^1 near solution

$$\mathbf{y} = A\mathbf{x} + \mathbf{e} \quad \text{s.t. } \|e\|_2 < \epsilon.$$



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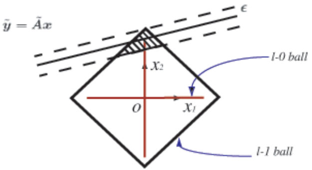

- **Bounded noise produces bounded ℓ^1 solution**

$$(P_1') \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \|\mathbf{y} - A\mathbf{x}\|_2 < \epsilon.$$

Restricted Isometry Property [Candès, Romberg, Tao 2004]: $\|\mathbf{x}^* - \mathbf{x}_0\|_2 < C\epsilon$.

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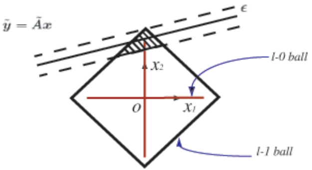

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- ℓ^1-minimization routines
  1. **Matching pursuit** [Mallat 1993]
  2. **Basis pursuit** [Chen 1998]
  3. **Lasso** [Tibshirani 1996]

Partial Features on Extended Yale B Database

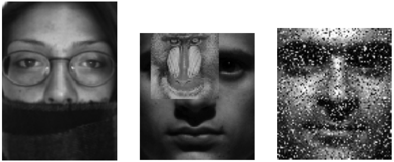
Features	Nose	Right Eye	Mouth & Chin
Dimension	4,270	5,040	12,936
SRC [%]	87.3	93.7	98.3
nearest-neighbor [%]	49.2	68.8	72.7
nearest-subspace [%]	83.7	78.6	94.4
Linear SVM [%]	70.8	85.8	95.3

SRC: sparse-representation classifier

Reference:

Robust face recognition via sparse representation, **(in press) PAMI**, 2008.

Occlusion Compensation



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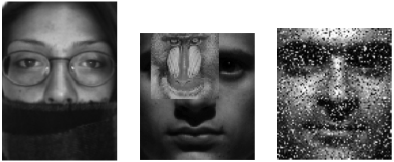
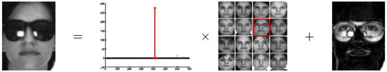


1. Sparse representation + sparse error

$$\mathbf{y} = A\mathbf{x} + \mathbf{e}$$



2. Occlusion compensation

$$\mathbf{y} = \begin{bmatrix} A & | & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = B\mathbf{w}$$

AR Database: 100 subjects, illumination, expression, occlusion

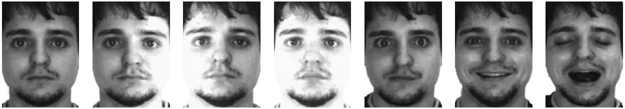
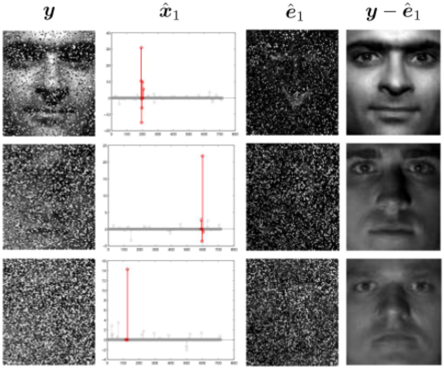
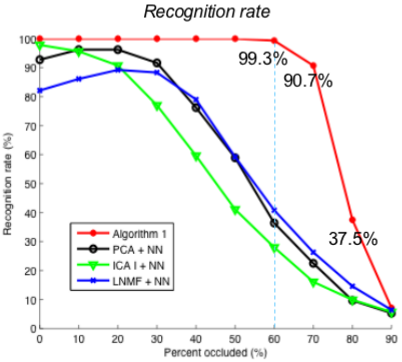


Figure: Training samples for Subject 1.

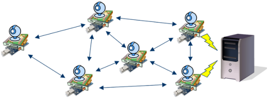
 illumination & expression	 sunglasses	 scarves
95%	97.5%	93.5%

Random Pixel Corruption





Future Direction: Distributed Sensor Perception (DSP)

Centralized Recognition	Distributed Recognition
	
powerful processors	mobile processors
(virtually) unlimited memory	limited onboard memory
(virtually) unlimited bandwidth	band-limited communications
simple sensor management	complex sensor networks

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	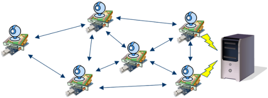
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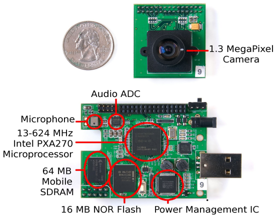
CITRIC: Wireless Smart Camera Sensor Platform

- CITRIC platform



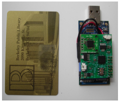
- Early adopters

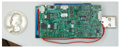
- A 3-second counter-sniper demo

CITRIC

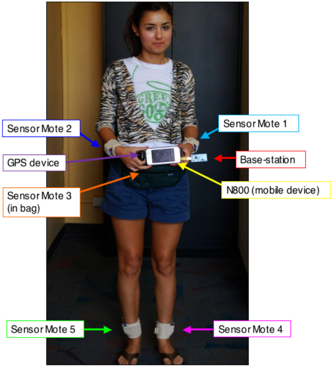
DexterNet: Wireless Body Sensor Network Platform

- Heterogeneous body sensors



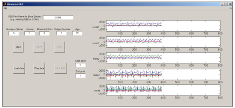


- Layout

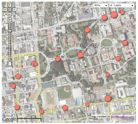


- Applications
  1. Wearable action recognition

  2. WARD database



  3. Long-term asthma study



Acknowledgments

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- **Cornell**: Philip Kuryloski
- **JHU**: Dr. René Vidal
- **UMich**: Dr. Harm Derksen
- **UT Dallas**: Dr. Roozbeh Jafari
- **Tampere University of Technology:** Ville-Pekka Seppa
- **Telecom Italia**: Dr. Marco Sgnoi, Roberta Giannantonio, Raffaele Gravina

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- ARO MURI: Heterogeneous Sensor Networks in Urban Terrains
- ARO MURI: Adaptive Coordinated Control of Intelligent Multi-Agent Teams
- NSF TRUST Center

References

- **SIAM Review:** Estimation of Subspace Arrangements with Applications in Modeling and Segmenting Mixed Data, 2008.
- **IJCV (draft):** Robust Algebraic Segmentation of Mixed Rigid-Body and Planar Motions, 2008.
- **CVIU:** Unsupervised Segmentation of Natural Images via Lossy Data Compression, 2008.
- **PAMI (in press):** Robust face recognition via sparse representation, 2008.
- **ICDSC:** CITRIC: A Low-Bandwidth Wireless Camera Network Platform, 2008.
- **IPSN (draft):** DexterNet: An open platform for heterogeneous body sensor networks and its applications, 2008.

ℓ^1-Minimization Routines

- **Matching pursuit** [Mallat 1993]
  1. Find most correlated vector $\mathbf{v}_i$ in A with $\mathbf{y}$: $i = \arg\max \langle \mathbf{y}, \mathbf{v}_j \rangle$.
  2. $A \leftarrow A^{(i)}$, $x_i \leftarrow \langle \mathbf{y}, \mathbf{v}_i \rangle$, $\mathbf{y} \leftarrow \mathbf{y} - x_i \mathbf{v}_i$.
  3. Repeat until $\|\mathbf{y}\| < \epsilon$.
- **Basis pursuit** [Chen 1998]
  1. Start with number of sparse coefficients $m = 1$.
  2. Select m linearly independent vectors B_m in A as a basis

$$\mathbf{x}_m = B_m^{\dagger} \mathbf{y}.$$

  3. Repeat swapping one basis vector in B_m with another vector not in B_m if improve $\|\mathbf{y} - B_m \mathbf{x}_m\|$.
  4. If $\|\mathbf{y} - B_m \mathbf{x}_m\|_2 < \epsilon$, stop; Otherwise, $m \leftarrow m + 1$, repeat Step 2.
- **Quadratic solvers**: $\mathbf{y} = A\mathbf{x}_0 + \mathbf{z} \in \mathbb{R}^d$, where $\|\mathbf{z}\|_2 < \epsilon$

$$\mathbf{x}^* = \arg\min\{\|\mathbf{x}\|_1 + \lambda \|\mathbf{y} - A\mathbf{x}\|_2\}$$

[LASSO, Second-order cone programming]: Much more expensive.

Matlab Toolboxes for ℓ^1-Minimization

- ℓ^1-**Magic** by Candes
- **SparseLab** by Donoho
- **cvx** by Boyd

Mild Conditions for ℓ^1/ℓ^0 Equivalence

$$(P_1) \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y} = A\mathbf{x}$$

Solve ℓ^1-minimization achieves the optimal sparse solution under the following conditions

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- Long answers
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$$\mu(A, B) \doteq \sup_{\mathbf{a}\in A, \mathbf{b}\in B} \frac{|\langle \mathbf{a}, \mathbf{b}\rangle|}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

$\|\mathbf{x}\|_0 \leq \frac{1}{2}(1 + \frac{1}{\mu(A,B)})$ suffices. A **and** B **have to be incoherent.**

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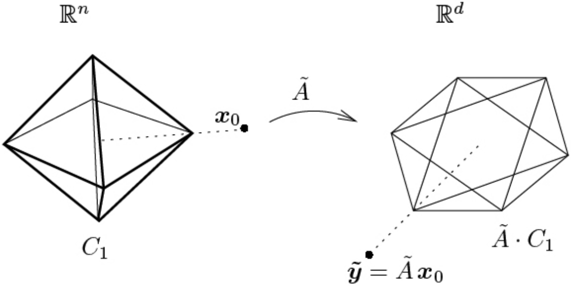
     $\|\mathbf{x}\|_0 \le \frac{1}{2}(1 + \frac{1}{\mu(A,B)})$ suffices. **A and B have to be incoherent.**
  2. **Restricted Isometry** [Candes & Tao 2005]:
     Define $\delta_k(A) \doteq min\delta$ such that

$$(1-\delta)\|\mathbf{x}\|_2^2 \le \|A\mathbf{x}\|_2^2 \le (1+\delta)\|\mathbf{x}\|_2^2 \quad \forall k\text{-sparse } \mathbf{x}.$$

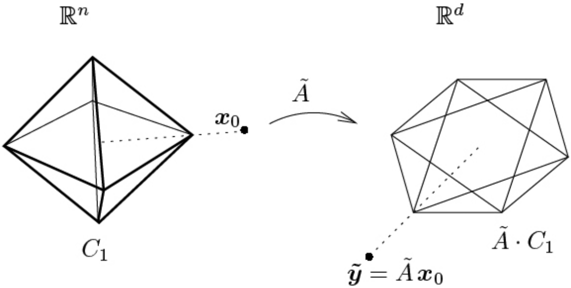
     $\delta_{2k}(A) \le \sqrt{2} - 1$ suffices. **The columns of A should be uniformly well-spread.**

k-Neighborlyness [Donoho 2006]



- Define **cross polytope** C and **quotient polytope** P such that $P = AC$.

k-Neighborlyness [Donoho 2006]



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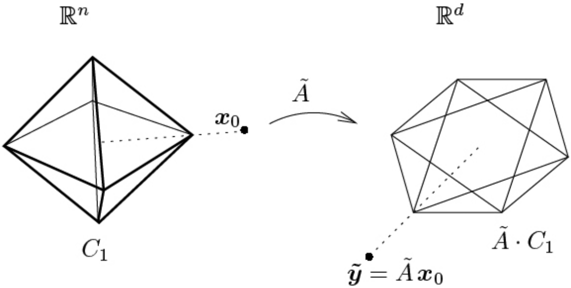

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- If $\mathbf{x}$ is k-sparse, $\mathbf{x}$ lie in a $(k-1)$-face of C in $\mathbb{R}^n$.
- **Necessary and Sufficient:** If ℓ^1/ℓ^0 holds for all k-sparse $\mathbf{x}$, all $(k-1)$-faces of C must be the faces of P on the boundary.

Sparse Representation in Classification: a Cross-and-Bouquet Model

- Traditional compressive sensing focuses on

$$\mathbf{y} = A\mathbf{x} + \mathbf{e}$$

1. A is component-wise Gaussian.
2. A is sparse Bernoulli.
3. A is megadictionary $[I|F]$, where F is Fourier or wavelets.

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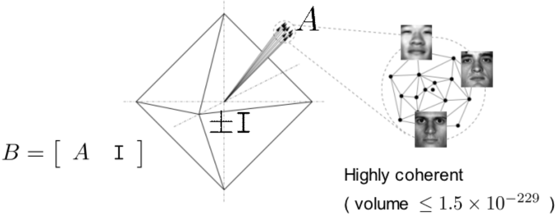
- Solving sparse representation for recognition purpose represents a special model

$$\mathbf{y} = [A \quad | \quad I]\begin{bmatrix}\mathbf{x}\\ \mathbf{e}\end{bmatrix}$$



Reference:

John Wright and Yi Ma, *Dense Error Correction via l1 Minimization*.