Homework Assignment 2 – 600.455/655 Fall 2020 (Circle One)

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment _____	Signature (required) I/We have followed the rules in completing this assignment _____

Note: 455 students may answer 655 questions for extra credit, but max total grade will still be 100

Question	Points (455)	Points (655)		Totals
1A	1	1		
1B	4	4		
1C	5	5		10
2A	5	5		
2B	5	5		
2C	11	5		
2D	12	10		
2E	12	10		
2F	0	5		40
3A	10	10		
3B	5	5		
3C	11	10		
3D	12	10		
3E	12	10		
3F	0	5		50
Total	100	100		

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. Remember to include a sealable 8 ½ by 11 inch self-addressed envelope if you want your assignment

Question 1

Suppose that we have $\mathbf{F}^* = \mathbf{F}\Delta\mathbf{F}_R = \Delta\mathbf{F}_L\mathbf{F}$, where

$$\mathbf{F} = [\mathbf{R}, \vec{\mathbf{p}}]$$

$$\Delta\mathbf{F}_L = [\Delta\mathbf{R}_L, \Delta\vec{\mathbf{p}}_L] \approx [\mathbf{I} + sk(\vec{\alpha}_L), \vec{\varepsilon}_L]$$

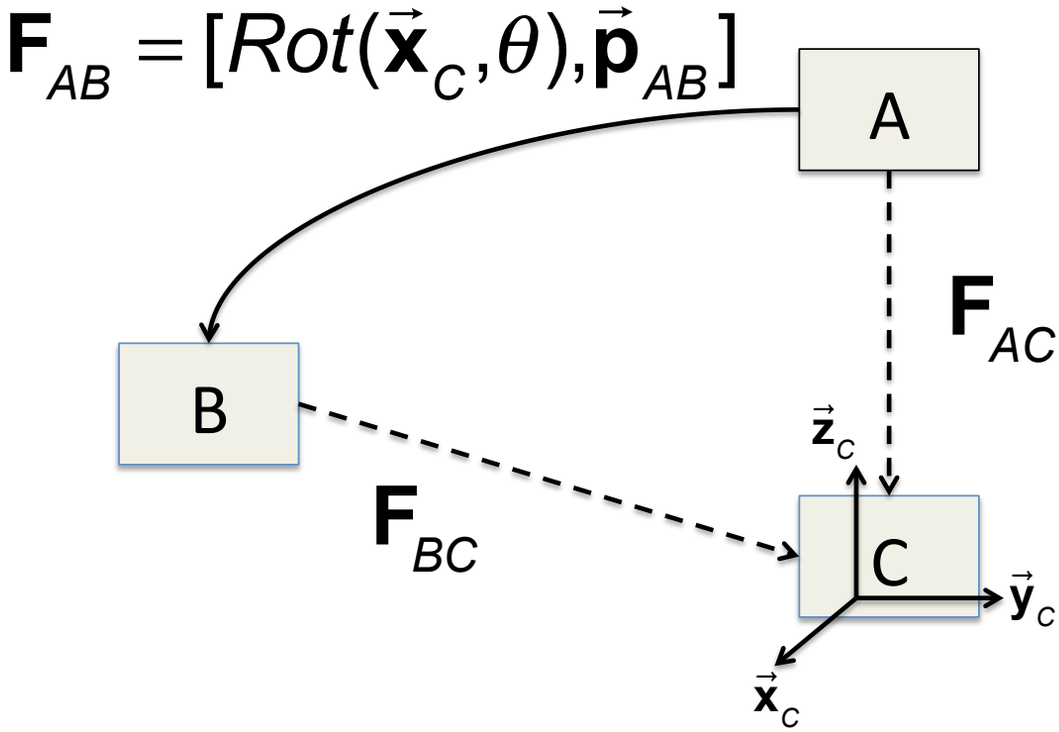
$$\Delta\mathbf{F}_R = [\Delta\mathbf{R}_R, \Delta\vec{\mathbf{p}}_R] \approx [\mathbf{I} + sk(\vec{\alpha}_R), \vec{\varepsilon}_R]$$

A. Give expressions for $\Delta\mathbf{F}_L$ and $\Delta\mathbf{F}_R$ in terms of $\mathbf{F}$, $\Delta\mathbf{F}_R$, and $\Delta\mathbf{F}_L$, avoiding tautologies like $\Delta\mathbf{F}_L = \Delta\mathbf{F}_L$.

B. Give expressions for $\Delta\mathbf{R}_L$, $\Delta\mathbf{R}_R$, $\Delta\vec{\mathbf{p}}_L$, and $\Delta\vec{\mathbf{p}}_R$ in terms of the other quantities while avoiding tautologies.

C. Give simplified expressions for the linearized error variables $\vec{\alpha}_L$, $\vec{\alpha}_R$, $\vec{\varepsilon}_L$, and $\vec{\varepsilon}_R$ in terms of the other quantities while avoiding tautologies. Also, express your answer in "standard" form, in which small error variables are shown as sums of terms with the general form $\mathbf{M}_k\vec{\eta}_k$ where $\vec{\eta}_k$ are small error variables and $\mathbf{M}_k$ is an expression containing quantities known to the computer. For example, one might imagine an answer $\vec{\gamma} = \mathbf{R}sk(\vec{\mathbf{v}})\vec{\alpha} + \vec{\beta}$.

Question 2



Consider the figure above. Here you can think of A and B as being some sort of pose sensors and C as being a target. The position and orientation of the two sensors is given by $\mathbf{F}_{AB} = [Rot(\vec{\mathbf{x}}_C, \theta), \vec{\mathbf{p}}_{AB}]$. I.e., the relative orientation of A and B is given by a rotation about the $\vec{\mathbf{x}}$ axis of C. $\mathbf{F}_{AC}$ and $\mathbf{F}_{BC}$ are the positions and orientations returned by the measurement sensors. Of course, there is some error, so that $\mathbf{F}_{AC}^{*} = \mathbf{F}_{AC}\Delta\mathbf{F}_{AC}$, where $\Delta\mathbf{F}_{AC} \approx [\mathbf{I} + sk(\vec{\alpha}_{AC}), \vec{\varepsilon}_{AC}]$ and similarly for $\mathbf{F}_{BC}^{*} = \mathbf{F}_{BC}\Delta\mathbf{F}_{BC}$ and $\mathbf{F}_{AB}^{*} = \mathbf{F}_{AB}\Delta\mathbf{F}_{AB}$, with $\Delta\mathbf{F}_{AB} \approx [\mathbf{I} + sk(\vec{\alpha}_{AB}), \vec{\varepsilon}_{AB}]$ and $\Delta\mathbf{F}_{BC} \approx [\mathbf{I} + sk(\vec{\alpha}_{BC}), \vec{\varepsilon}_{BC}]$.

Suppose that we have some information about the sensor uncertainties. In particular, the sensors A and B may be less accurate in "depth" than they are laterally. This is expressed component wise as

$\left|\vec{\varepsilon}_{AC}\right| \leq [\eta,\eta,\nu]^T$ (i.e., $\left|\varepsilon_{AC,x}\right| \leq \eta$, $\left|\varepsilon_{AC,y}\right| \leq \eta$, $\left|\varepsilon_{AC,z}\right| \leq \nu$), and $\eta \leq \nu$

$\left|\vec{\alpha}_{AC}\right| \leq [\phi,\phi,\phi]^T$

$\left|\vec{\alpha}_{AB}\right| \leq [\rho,\rho,\rho]^T \quad \left|\vec{\varepsilon}_{AB}\right| \leq [\sigma,\sigma,\sigma]$

$\left|\vec{\varepsilon}_{BC}\right| \leq [\eta,\eta,\nu]^T \quad \left|\vec{\alpha}_{BC}\right| \leq [\phi,\phi,\phi]^T$

Otherwise, we know that the errors are "small". Clearly, the direct measurement $\mathbf{F}_{AC}$ can be combined with an additional measurement computed from $\mathbf{F}_{AB}$ and $\mathbf{F}_{BC}$ to produce a new estimated value $\mathbf{F}_{AC}^{est}$ for $\mathbf{F}_{AC}^{*}$. This new estimate will have an associated error $\mathbf{F}_{AC}^{*} = \mathbf{F}_{AC}^{est}\Delta\mathbf{F}_{AC}^{est}$, where $\Delta\mathbf{F}_{AC}^{est} \approx [\mathbf{I} + sk(\vec{\alpha}_{AC}^{est}), \vec{\varepsilon}_{AC}^{est}]$.

A. Write down the components of $\mathbf{F}_{ABC} = [\mathbf{R}_{ABC}, \vec{\mathbf{p}}_{ABC}]$ based on the kinematic calculation $\mathbf{F}_{ABC} = \mathbf{F}_{AB}\mathbf{F}_{BC}$.

- **Notational Hint:** Please give expressions for $\mathbf{R}_{ABC}$ and $\vec{\mathbf{p}}_{ABC}$. We are adding this notation to the problem to simplify the grading and also to help clarify the question and approach. In Question 1D,

you will take advantage of the fact that there may also be a direct measurement of $\mathbf{F}_{AC}$.

B. Write down the components of $\Delta\mathbf{F}_{ABC} = [\Delta\mathbf{R}_{ABC}, \Delta\vec{\mathbf{p}}_{ABC}]$ based on the kinematic calculation $\mathbf{F}_{ABC} = \mathbf{F}_{AB}\mathbf{F}_{BC}$. Express your answer in terms of the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s.

C. Write down the linearized approximation components $[\vec{\alpha}_{ABC}, \vec{\varepsilon}_{ABC}]$ $\Delta\mathbf{F}_{ABC} \approx [\mathbf{I} + sk(\vec{\alpha}_{ABC}), \vec{\varepsilon}_{ABC}]$. Express your answer in standard linearized form.

D. **NOTE: I have produced a simplified version of this problem from the previous version.** Assume that you have been given a different model for the sensor errors:

$$\mathbf{F}^*_{AC} = \Delta\mathbf{F}^L_{AC}\mathbf{F}_{AC} \qquad \Delta\mathbf{R}^L_{AC} \approx \mathbf{I} + sk(\vec{\alpha}^L_{AC}) \quad \Delta\vec{\mathbf{p}}^L_{AC} \approx \vec{\varepsilon}^L_{AC}$$
$$\mathbf{F}^*_{BC} = \Delta\mathbf{F}^L_{BC}\mathbf{F}_{BC} \qquad \Delta\mathbf{R}^L_{BC} \approx \mathbf{I} + sk(\vec{\alpha}^L_{BC}) \quad \Delta\vec{\mathbf{p}}^L_{BC} \approx \vec{\varepsilon}^L_{BC}$$

Suppose also that we know that

$\left|\vec{\varepsilon}^L_{AC}\right| \le [\eta, \eta, \nu]^T$ (i.e., $\left|\varepsilon^L_{AC,x}\right| \le \eta$, $\left|\varepsilon^L_{AC,y}\right| \le \eta$, $\left|\varepsilon^L_{AC,z}\right| \le \nu$), and $\eta \le \nu$

$\left|\vec{\alpha}^L_{AC}\right| \le [\phi, \phi, \phi]^T$

$\left|\vec{\alpha}_{AB}\right| \le [\rho, \rho, \rho]^T \quad \left|\vec{\varepsilon}_{AB}\right| \le [\sigma, \sigma, \sigma]$

$\left|\vec{\varepsilon}^L_{BC}\right| \le [\eta, \eta, \nu]^T \quad \left|\vec{\alpha}^L_{BC}\right| \le [\phi, \phi, \phi]^T$

We thus have two ways to estimate the quantity

$$\mathbf{F}^*_{AC} = \Delta\mathbf{F}^L_{AC}\mathbf{F}_{AC} = \mathbf{F}_{AB}\Delta\mathbf{F}_{AB}\Delta\mathbf{F}^L_{BC}\mathbf{F}_{BC}$$

All this information can be combined to produce an improved estimate $\mathbf{F}_{AC}^{*} = \Delta\mathbf{F}_{AC}^{est}\mathbf{F}_{AC}^{est}$ of C relative to A with

$$\Delta\mathbf{F}_{AC}^{est} \approx [\mathbf{I} + sk(\vec{\alpha}_{AC}^{est}), \vec{\varepsilon}_{AC}^{est}].$$

Given some arbitrary direction given by a unit vector $\vec{\mathbf{d}}$, then the positional error in direction $\vec{\mathbf{d}}$ will be given by $\xi = \vec{\mathbf{d}} \bullet \vec{\varepsilon}_{AC}^{est}$ subject to a set of constraints. Write down a linearized set of constraints in standard form that may be used to find a limit on the maximum value of ξ .

Note that this will include what you know about the measurement uncertainties.

ONE other note: For the purpose of this assignment, you can assume that the values returned by the sensors are compatible. I.e., $\mathbf{F}_{AC} = \mathbf{F}_{AB}\mathbf{F}_{BC}$, even though the errors will be different. In reality, this won't be the case, and the analysis will be harder. But the simplification will make the problem a bit easier.

E. **Using the assumptions in Question 2D**, suppose now that we have the following additional information

$$\eta = \sigma = 0.5 \text{ mm}, \ \nu = 5 \text{ mm}, \ \phi = \rho = 0.00001 \text{ radians}$$

$$\theta = 90^{\circ}, \ \vec{\mathbf{p}}_{AB} = Rot(\vec{\mathbf{x}}, \theta) \bullet [0,0, 500 \text{ mm}]^{T} - [0,0,500 \text{ mm}]$$

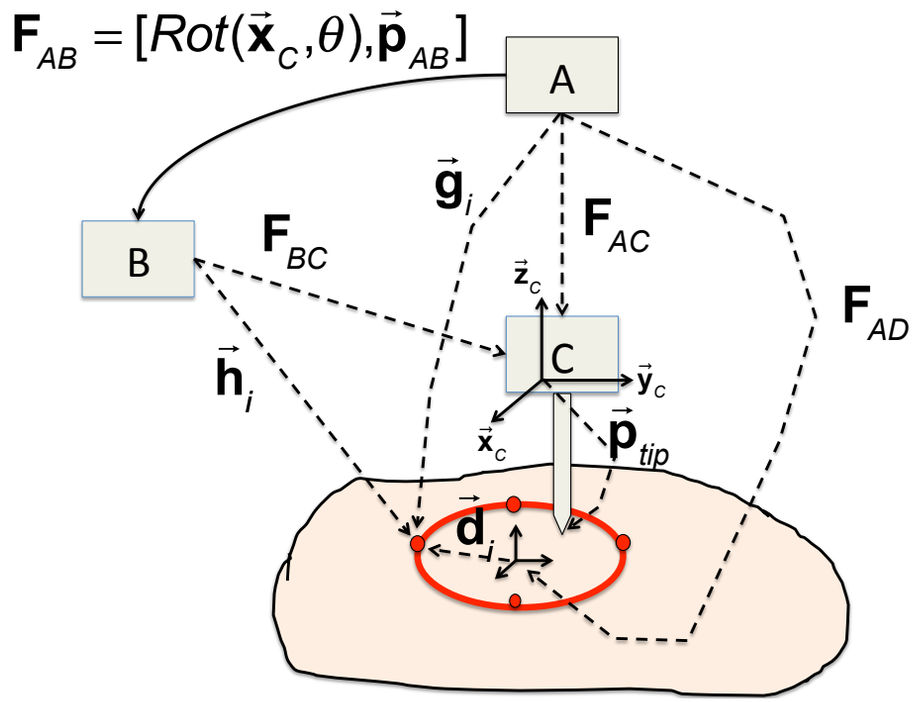
$$\vec{\mathbf{p}}_{AC} = [0,0, -500 \text{ mm}]^{T}$$

Give an estimate on the limits of $\vec{\varepsilon}_{AC}^{est}$ to within approximately 0.1 mm.

Hint: Make a sketch and think some before you do a great deal of calculating.

F. (600.655 only) Under the same accuracy assumptions as above, estimate what is the smallest value of θ for which you can guarantee that $\left|\vec{\varepsilon}_{AC}{}^{est}\right| \leq 3\,\text{mm}$?

Question 3



Consider now the situation in the figure above, which adds to that used for Question 2. The target body C has been attached to a pointer, and a calibration has been performed to compute the position $\vec{\mathbf{p}}_{tip}$ of the pointer tip relative to the coordinate system of C. In addition, a fiducial structure D has been attached to the patient's body. This fiducial object comprises a number of point fiducials at locations $\vec{\mathbf{d}}_i$ relative to the coordinate system of D. The sensor A senses these point fiducials and reports that the location of the i'th fiducial is at location $\vec{\mathbf{g}}_i$ relative to the coordinate system of A. Similarly, sensor B reports that that the location of the i'th fiducial is at location $\vec{\mathbf{h}}_i$ relative to the coordinate system of B. There is, of course, some error:

$$\vec{\mathbf{p}}_{tip}{}^{*} = \vec{\mathbf{p}}_{tip} + \vec{\varepsilon}_{tip} \quad \text{where } \left\|\vec{\varepsilon}_{tip}\right\| \le \psi$$

$$\vec{\mathbf{d}}_{i}{}^{*} = \vec{\mathbf{d}}_{i} + \vec{\varepsilon}_{D,i} \quad \text{where } \left\|\vec{\varepsilon}_{D,i}\right\| \le \delta$$

$$\vec{\mathbf{g}}_{i}{}^{*} = \vec{\mathbf{g}}_{i} + \vec{\varepsilon}_{A,i} \quad \text{where } \left|\vec{\varepsilon}_{A,i}\right| \le [\eta,\eta,\nu]^{T}$$

$$\vec{\mathbf{h}}_{i}{}^{*} = \vec{\mathbf{h}}_{i} + \vec{\varepsilon}_{B,i} \quad \text{where } \left|\vec{\varepsilon}_{B,i}\right| \le [\eta,\eta,\nu]^{T}$$

Assume that you have software that computes $\mathbf{F}_{AC}{}^{est}$, with $\mathbf{F}_{AC}{}^{*} = \mathbf{F}_{AC}{}^{est}\Delta\mathbf{F}_{AC}{}^{est}$ and $\Delta\mathbf{F}_{AC}{}^{est} \approx [\mathbf{I} + sk(\vec{\alpha}_{AC}{}^{est}), \vec{\varepsilon}_{AC}{}^{est}]$. Further, we have the following nominal values:

$$\vec{\mathbf{d}}_1 = [0,50,0]^T \qquad \vec{\mathbf{g}}_1 = [0,50,-650]^T$$

$$\vec{\mathbf{d}}_2 = [50,0,0]^T \qquad \vec{\mathbf{g}}_2 = [50,0,-650]^T$$

$$\vec{\mathbf{d}}_3 = [0,-50,0]^T \qquad \vec{\mathbf{g}}_3 = [0,-50,-650]^T$$

$$\vec{\mathbf{d}}_3 = [-50,0,0]^T \qquad \vec{\mathbf{g}}_4 = [-50,0,-650]^T$$

A. Since we have redundant information, the A and B sensor values may be combined to produce an estimate $\vec{\mathbf{g}}_i^{est}$ of the position of the i'th fiducial relative to sensor A. Suppose that we know that $\rho = \sigma = 0$, so that the transformation $\mathbf{F}_{AB}$ is known exactly as a function of θ. Under the assumptions of Questions 2A-2D, provide a set of constraints in standard form constraining how

accurately you determine $\vec{\mathbf{g}}_i^{est}$? I.e., how can you constrain $\vec{\varepsilon}_i^{est}$, where $\vec{\mathbf{g}}_i^{est*} = \vec{\mathbf{g}}_i^{est} + \vec{\varepsilon}_i^{est}$?

B. Assuming that you have a correct algorithm for computing $\mathbf{F}_{AD}$ from the $\vec{\mathbf{g}}_i^{est}$, give a formula for computing the position $\vec{\mathbf{p}}_{Dt}$ of the pointer tip relative to the coordinate system of D.

C. Assume now that (due to our superior analysis and possible invocation of an oracle) we can assume that $\left|\vec{\varepsilon}_i^{est}\right| \le \left[\delta, \delta, \psi\right]^T$. How accurately can you determine $\mathbf{F}_{AD}$? I.e., can you express some constraints or limits on $\Delta\mathbf{F}_{AD} = [\mathbf{I} + sk(\vec{\alpha}_{AD}), \vec{\varepsilon}_{AD}]$?

D. Write a formula giving limits for the error $\vec{\varepsilon}_{Dt}$ in $\vec{\mathbf{p}}_{Dt}$, under the assumptions of Questions 3A-3C.

E. Suppose that we know that $\mathbf{F}_{AC}^{est} = [\mathbf{I}, [0,0,-500]^T]$, $\vec{\mathbf{p}}_{tip} = [0,0,-100]^T$, and $\psi = \delta = 0.3$ mm. Suppose that we know further that $\|\vec{\alpha}_{AC}\|_2 \le \gamma$, with $\gamma = 0.001$ radians. Estimate a conservative numerical estimate (to ~0.1mm precision) limit on $\left\|\vec{\varepsilon}_{Dt}\right\|_2$.

F. (655 only) Under the assumptions above, what value of $\delta = \psi$ would be required in order to ensure that $\left\|\vec{\varepsilon}_{Dt}\right\|_2 \le 1.5$ mm? (You can round off to nearest 0.05 mm)