Homework Assignment 3 – 600.445 Fall 2019

Instructions and Score Sheet (hand in with answers)

Name	Name
Email	Email
Other contact information (optional)	Other contact information (optional)
Signature (required) I/We have followed the rules in completing this assignment ______________________________	Signature (required) I/We have followed the rules in completing this assignment ______________________________

Question	455 Points	655 Points	Points	Totals
A	10	10		
B	10	10		
C	15	15		
D	15	15		
E	10	10		
F	20	20		
G	10	10		
H	10	10		
I	10	10		
J	10	10		
Total	120	120		

Note: Although there are 120 points possible from the questions, at most 100 points will be counted toward your course grade.

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. Remember to include a sealable 8 ½ by 11 inch self-addressed envelope if you want your assignment

Question

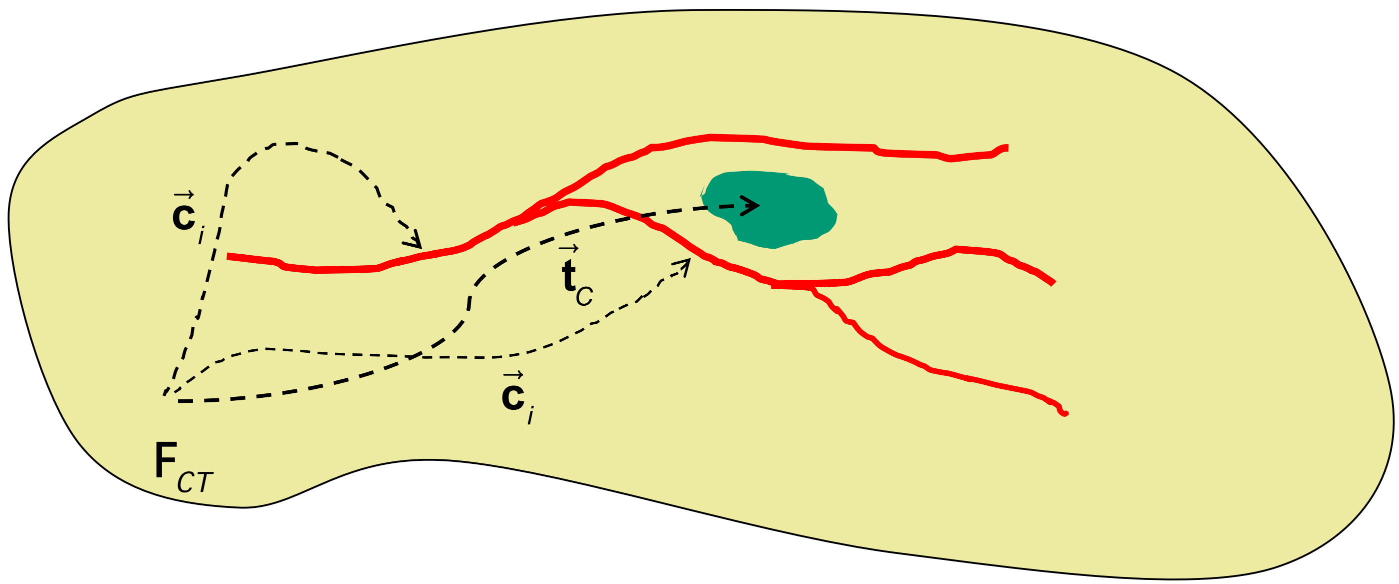
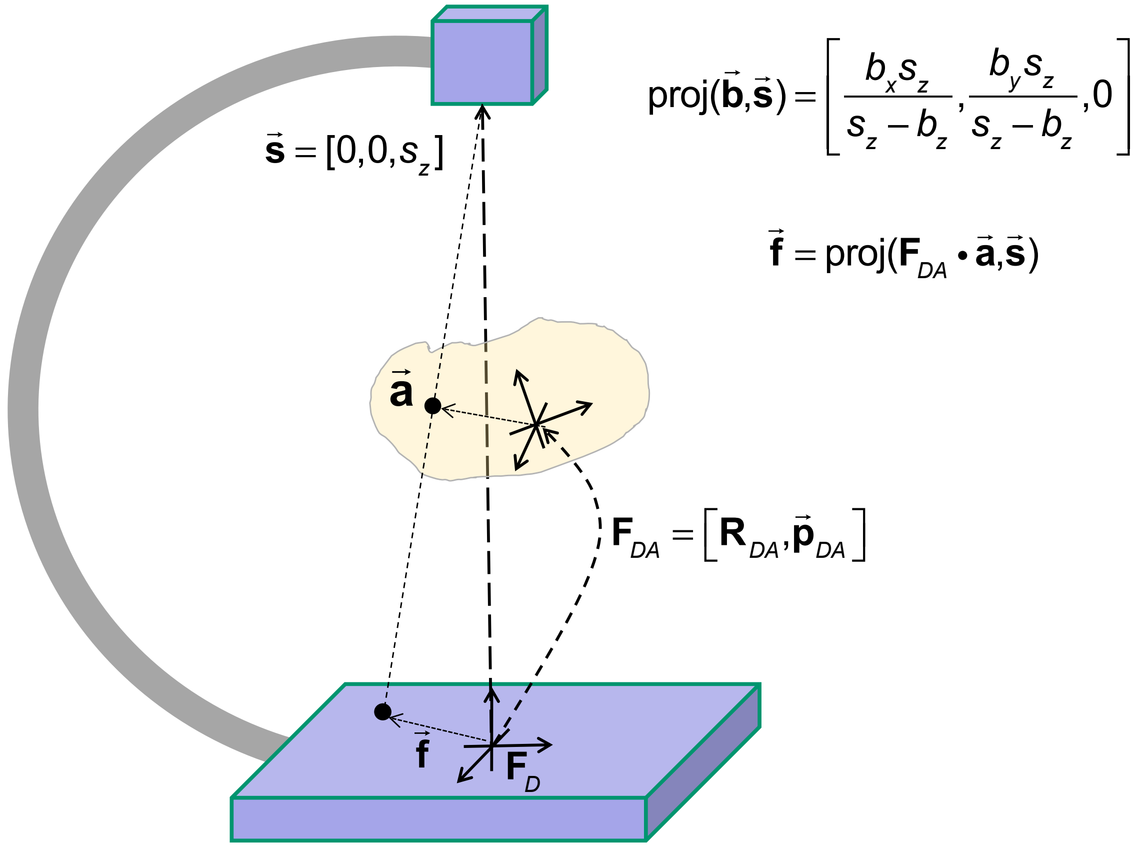


Fig. 1: Segmented CT scan of patient with tumor and vascular structures

Suppose that a patient has a tumor that is visible in CT, as indicated in Fig. 1. This tumor is located near some vascular structures that can be located in the same CT image after injection of suitable contrast material. A tumor biopsy site $\vec{\mathbf{t}}_C$ has been selected in the CT coordinates and the vascular structures have been segmented to produce a number of points $\vec{\mathbf{c}}_i$

corresponding to the vascular structures have been segmented in the CT images.

In the operating room, a flat panel x-ray c-arm is available, as shown in Fig. 2. The position of the source in detector coordinates is given by $\vec{\mathbf{s}}$ and there is no distortion.



$\text{proj}(\vec{\mathbf{b}},\vec{\mathbf{s}}) = \left[\frac{b_x s_z}{s_z - b_z}, \frac{b_y s_z}{s_z - b_z}, 0 \right]$
$\vec{\mathbf{s}} = [0,0,s_z]$
$\vec{\mathbf{f}} = \text{proj}(\mathbf{F}_{DA} \bullet \vec{\mathbf{a}}, \vec{\mathbf{s}})$
$\vec{\mathbf{a}}$
$\mathbf{F}_{DA} = \left[\mathbf{R}_{DA}, \vec{\mathbf{p}}_{DA} \right]$
$\vec{\mathbf{f}}$
$\mathbf{F}_D$

Fig. 2

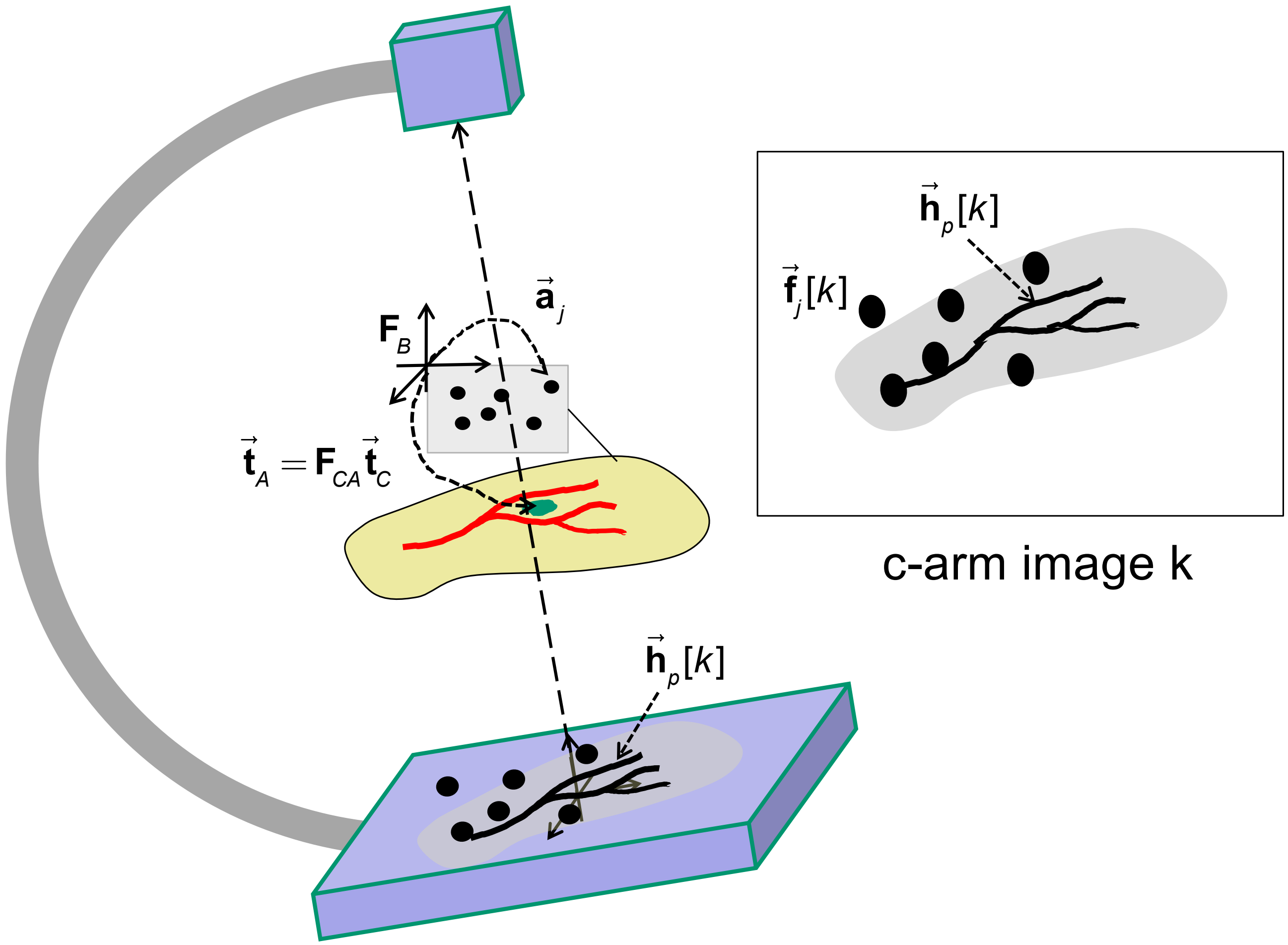

$\vec{\mathbf{a}}_j$
$\mathbf{F}_B$
$\vec{\mathbf{t}}_A = \mathbf{F}_{CA}\vec{\mathbf{t}}_C$
$\vec{\mathbf{h}}_p[k]$
$\vec{\mathbf{f}}_j[k]$
$\vec{\mathbf{h}}_p[k]$
c-arm image k

Fig. 3

A fiducial structure has been attached to the patient in the operating room. The structure is radiolucent but has embedded in it a moderate number N_{markers} (about 8) of small opaque spheres at positions $\vec{\mathbf{a}}_j$ relative to the fiducial structure. The c-arm may be positioned at a number of different poses relative to the patient and images showing the spheres and the vascular structure may be obtained. For pose k the image coordinates for sphere $\mathbf{a}_j$ are given by $\vec{\mathbf{f}}_j[k]$. There is image processing software available for determining these positions, though not necessarily for determining which image spot corresponds to which sphere. However, it is known that the spheres are arranged so that no two poses will produce the same projection pattern in the x-ray images. There is image processing software available to find the centerlines of the vascular structures. These are available as a large set of sample points $\vec{\mathbf{h}}_p[k]$ that are on the centerlines of the images of the structures. This software produces no correspondence between the $\vec{\mathbf{h}}_p[k]$ any particular vascular structure. However, it is safe to assume that any point $\vec{\mathbf{h}}_p[k]$ is the projected image of some centerline point $\vec{\mathbf{c}}_i$ on some vascular structure.

The image processing software has also located a tumor within the image and a biopsy target $\vec{\mathbf{t}}_C$ and entry point $\vec{\mathbf{t}}_E$ have been determined in CT coordinates.

A. Consider a point with coordinates $\vec{\mathbf{b}} = [b_x, b_y, b_z]$ relative to the c-arm detector and its projection $\vec{\mathbf{f}} = [u, v, 0]$ on the detector. Suppose that the point is moved by a small amount $\Delta\vec{\mathbf{b}} = [\Delta b_x, \Delta b_y, \Delta b_z]$, produce formulas linear in the elements of $\Delta\mathbf{b}$ to estimate the change in coordinates $\Delta\vec{\mathbf{f}} = [\Delta u, \Delta v, 0]$ of the projection.

B. Suppose that the fiducial structure moves slightly relative to the c-arm by an amount $\Delta\mathbf{F}_B \approx [\mathbf{I} + sk(\alpha_B), \vec{\varepsilon}_B]$, where $\mathbf{F}_B = [\mathbf{R}_B, \vec{\mathbf{p}}_B]$ is the previous pose of the fiducial structure. Given a marker position $\vec{\mathbf{b}}_j + \Delta\vec{\mathbf{b}}_j = \mathbf{F}_B \Delta\mathbf{F}_B \vec{\mathbf{a}}_j$ on the fiducial structure, with corresponding image projection $\vec{\mathbf{f}}_j + \Delta\vec{\mathbf{f}}_j = proj(\vec{\mathbf{b}}_j + \Delta\vec{\mathbf{b}}, \vec{\mathbf{s}})$, produce formulas for $\Delta\vec{\mathbf{f}}_j = [\Delta u_j, \Delta v_j, 0]$ in terms of $\vec{\alpha}_B$ and $\vec{\varepsilon}_B$. Express these results as a linearized matrix express with the general format

$$\Delta\vec{\mathbf{f}}_j = \begin{bmatrix} \Delta u_j \\ \Delta v_j \end{bmatrix} = \mathbf{M}_j \bullet \vec{\phi}_B \quad \text{where } \vec{\phi}_B = \begin{bmatrix} \vec{\alpha}_B \\ \vec{\varepsilon}_B \end{bmatrix}$$

Hint: Produce formulas to estimate the change in coordinates for $\Delta\vec{\mathbf{b}}_j$ and use the results from Question A.

C. Given a collection of projection values $\vec{\mathbf{f}}_j$ for markers $\vec{\mathbf{a}}_j$ embedded in the fiducial structure, suppose that the computer has used an available algorithm to compute an estimated pose $\mathbf{F}_B = [\mathbf{R}_B, \vec{\mathbf{p}}_B]$ relative to the c-arm detector for the fiducial structure, so that a point $\vec{\mathbf{a}}$ defined relative to the fiducial coordinate system has coordinates $\vec{\mathbf{b}} = \mathbf{F}_B\vec{\mathbf{a}}$ relative to the detector. However, the image acquisition hardware and image processing software is not perfect, so that there is some error $\Delta\vec{\mathbf{f}}_j$ in determining the projection value for each marker. Recall that the $\vec{\mathbf{f}}_j$ values will be used to determine the estimated pose. The errors $\Delta\vec{\mathbf{f}}_j$ will introduce some error in the calculation of $\mathbf{F}_B$. For a given feature with a location $\vec{\mathbf{c}}$ defined relative to the fiducial structure how you would estimate bounds on the error in your computation of $\mathbf{F}_B\vec{\mathbf{c}}$, given bounds $\left\|\Delta\vec{\mathbf{f}}_j\right\| \leq \rho$. Here, I would like you to explain how to compute a bound on the error in an arbitrary direction $\vec{\mathbf{d}}$, where $\left\|\vec{\mathbf{d}}\right\| = 1$. Would you expect the error bounds to be the same in all directions? If not, is there a particular direction that is likely to have worse error? Which direction?

You can assume that you have available a linear programming system capable of solving problems of the general form:

$$\zeta = \underset{\mathbf{x}}{\arg\min}\ \mathbf{c} \bullet \mathbf{x} \quad \text{subject to } \mathbf{Ax} \leq \mathbf{b}$$

Note: Here **A** is an arbitrary $m \times n$ matrix, the symbols **c** and **x** refer to vectors with n elements and **b** is a vector with n elements. These symbols have no relationship to the $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ symbols use in the problem scenario.

D. Suppose now that we want to implement a registration scheme in which we wish to use two x-ray images taken from different viewing directions, but with the fiducial structure and anatomy within the field of view of the c-arm. Outline a method for estimating accuracy of the computer's estimate the relative transformation between the two viewing positions under the same assumptions as in Question C. I.e., give formulas for $\vec{\phi}_{12} = [\vec{\alpha}_{12}{}^T, \vec{\varepsilon}_{12}{}^T]^T$, where $\mathbf{F}_{12} = \mathbf{F}_{B,1}{}^{-1}\mathbf{F}_{B,2}$ and

$$\mathbf{F}_{12}\Delta\mathbf{F}_{12} = \left(\mathbf{F}_{B,1}\Delta\mathbf{F}_{B,1}\right)^{-1}\mathbf{F}_{B,2}\Delta\mathbf{F}_{B,2}$$

$$\Delta\mathbf{F}_{12}(\vec{\phi}_{12}) \approx [\mathbf{I} + sk(\vec{\alpha}_{12}), \vec{\varepsilon}_{12}]$$

$$\Delta\mathbf{F}_{B,1}(\vec{\phi}_{B,1}) \approx [\mathbf{I} + sk(\vec{\alpha}_{B,1}), \vec{\varepsilon}_{B,1}]$$

$$\Delta\mathbf{F}_{B,2}(\vec{\phi}_{B,2}) \approx [\mathbf{I} + sk(\vec{\alpha}_{B,2}), \vec{\varepsilon}_{B,2}]$$

E. Under the assumptions above, how would you estimate bounds on the elements of $\vec{\phi}_{12} = [\vec{\alpha}_{12}{}^T, \vec{\varepsilon}_{12}{}^T]^T$?

F. Outline a method for determining the registration transformation $\mathbf{F}_{CA}$ from CT coordinates to the coordinate system of the fiducial structure. I.e., if an anatomic feature has location $\vec{\mathbf{c}}$ in CT coordinates, the point will have coordinates $\mathbf{F}_{CA}\vec{\mathbf{c}}$ relative to the fiducial structure. Here, I am looking for a method that would "work" for any number of images, though it will obviously be more accurate if several images are taken from different directions. Here, you can refer back to the answers to previous questions without reciting them again. Also, you can assume that an approximate initial guess for $\mathbf{F}_{CA}$ is available.

G. Suppose that the method developed in Question F has some small error, so that the actual transformation is given by:

$$\mathbf{F}_{CA}{}^* = \Delta\mathbf{F}_{CA}\mathbf{F}_{CA} \quad \text{(note order here)}$$

$$\Delta\mathbf{F}_{CA} \approx [\mathbf{I} + sk(\vec{\alpha}_{CA}), \vec{\varepsilon}_{CA}]$$

Suppose, further, that you have access to a very accurate needle guidance system that has been accurately co-registered to the fiducial structure. However, the small registration error will produce some placement errors for the needle. If the targeted position relative to the fiducial system is $\mathbf{F}_{CA}\vec{\mathbf{t}}_C$ then the actual position of the needle tip in CT coordinates when the biopsy is performed will be $\vec{\mathbf{t}}_C^{\,*} = \vec{\mathbf{t}}_C + \vec{\varepsilon}_C$. Give an expression for the approximate value of $\vec{\varepsilon}_C$.

H. Suppose that the component of $\vec{\varepsilon}_C$ along the needle path is large and there is an anatomic structure just beyond the tumor. This can often happen if only one image is used for registration and the needle path is generally along the registration direction. Is there something you can do to still provide a reasonably accurate biopsy that doesn't send a needle beyond the tumor? If so, what? Here, you can assume that the needle is to be inserted generally along the direction from the x-ray source to the detector. **Hint:** This will not require fancy mathematics, just some simple geometric reasoning and possibly some very simple hardware.

I. In addition to the assumptions in Question G, suppose that it is possible to place the needle tip onto the entry point $\vec{\mathbf{t}}_E$ with very high precision. For example, a radiopaque skin marker may have been placed at the entry point before the CT scan was taken. How will this affect the accuracy with which you can perform the biopsy?

J. In addition to the assumptions in Question I, assume further that you have bounds on the magnitude of the registration error. I.e.,

$$\left\| \vec{\alpha}_{CA} \right\| \leq \gamma$$

$$\left\| \vec{\varepsilon}_{CA} \right\| \leq \delta$$

How will this affect the accuracy with which you can place the needle on the biopsy target?