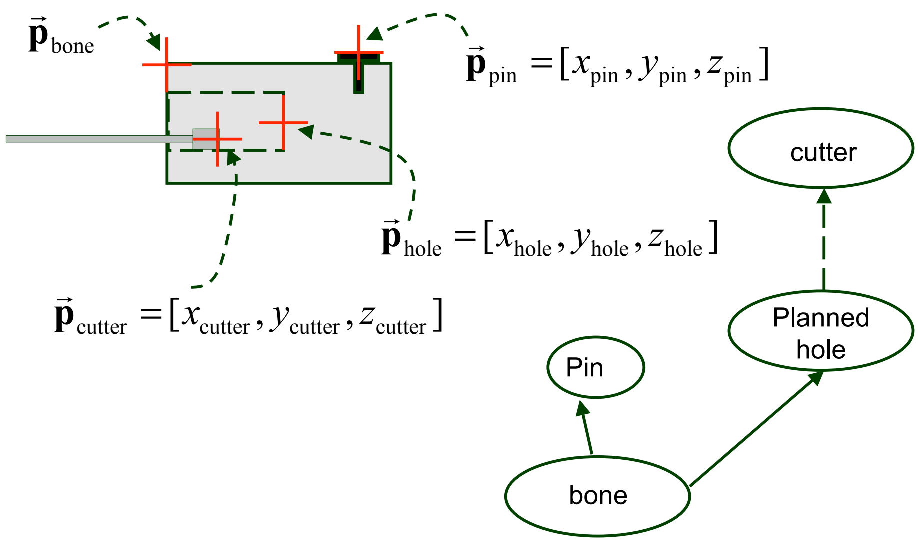
Introduction to Vectors and Frames
A simplified view

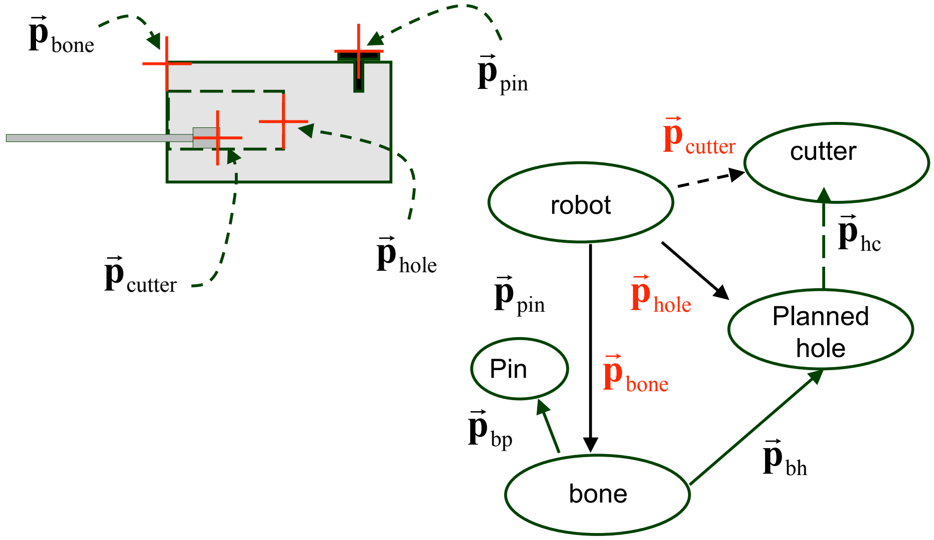
CIS - 600.445

Russell Taylor

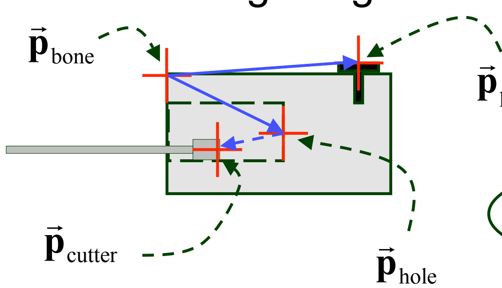
Defining things relative to other things



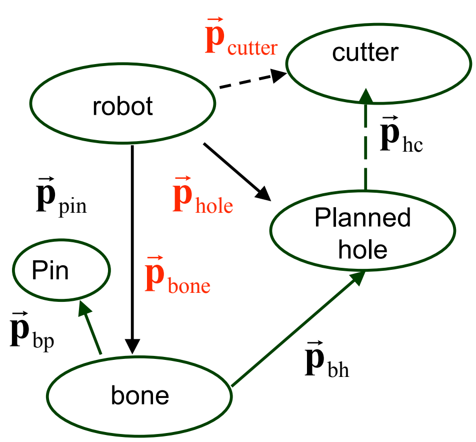
Defining things relative to other things



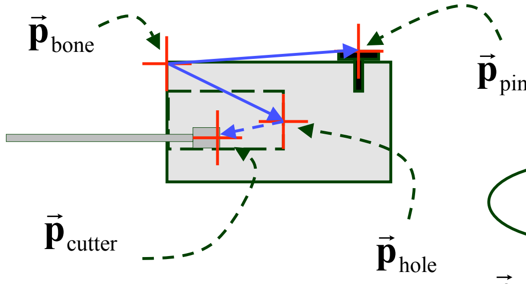
Defining things relative to other things

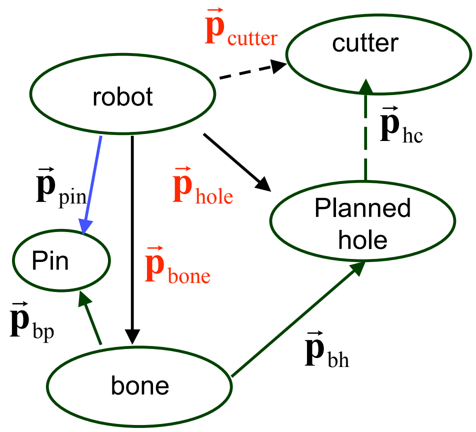


$$\vec{\mathbf{p}}_{\text{cutter}} = \vec{\mathbf{p}}_{\text{hole}} + \vec{\mathbf{p}}_{\text{hc}} = \begin{bmatrix} x_{\text{hole}} + x_{\text{hc}} \\ y_{\text{hole}} + y_{\text{hc}} \\ z_{\text{hole}} + z_{\text{hc}} \end{bmatrix}$$



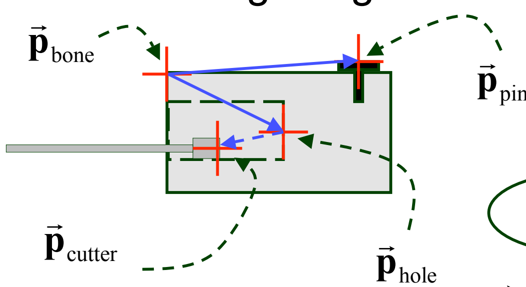
Defining things relative to other things

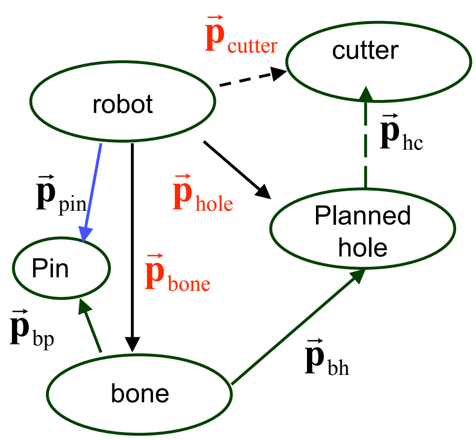




$$\vec{p}_{\text{cutter}} = \vec{p}_{\text{hole}} + \vec{p}_{\text{hc}}$$
$$\vec{p}_{\text{hole}} = \vec{p}_{\text{bone}} + \vec{p}_{\text{bh}}$$
$$\vec{p}_{\text{pin}} = \vec{p}_{\text{bone}} + \vec{p}_{\text{bp}}$$

Defining things relative to other things





$$\vec{p}_{\text{cutter}} = \vec{p}_{\text{hole}} + \vec{p}_{\text{hc}}$$
$$\vec{p}_{\text{hole}} = \vec{p}_{\text{bone}} + \vec{p}_{\text{bh}}$$
$$\vec{p}_{\text{pin}} = \vec{p}_{\text{bone}} + \vec{p}_{\text{bp}}$$

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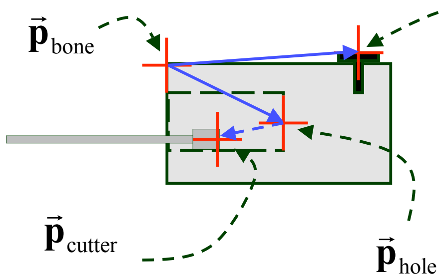
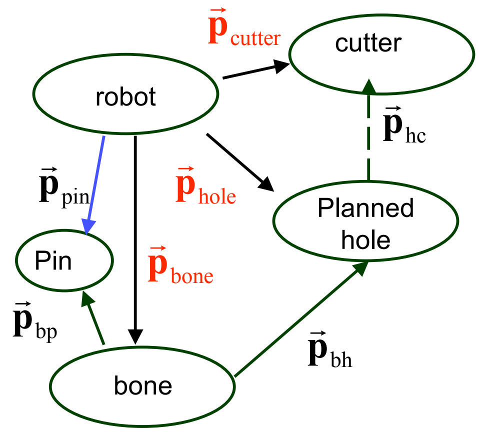
Defining things relative to other things



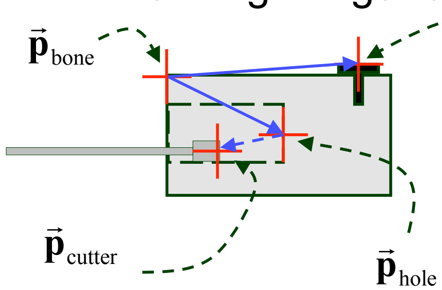


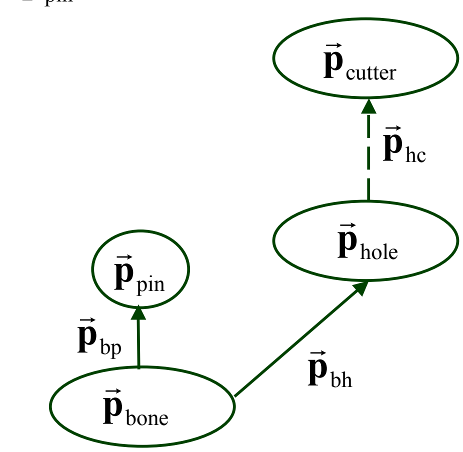
$$\vec{p}_{bone} = \vec{p}_{pin} - \vec{p}_{bp}$$

$$\vec{p}_{hole} = \vec{p}_{bone} + \vec{p}_{bh}$$

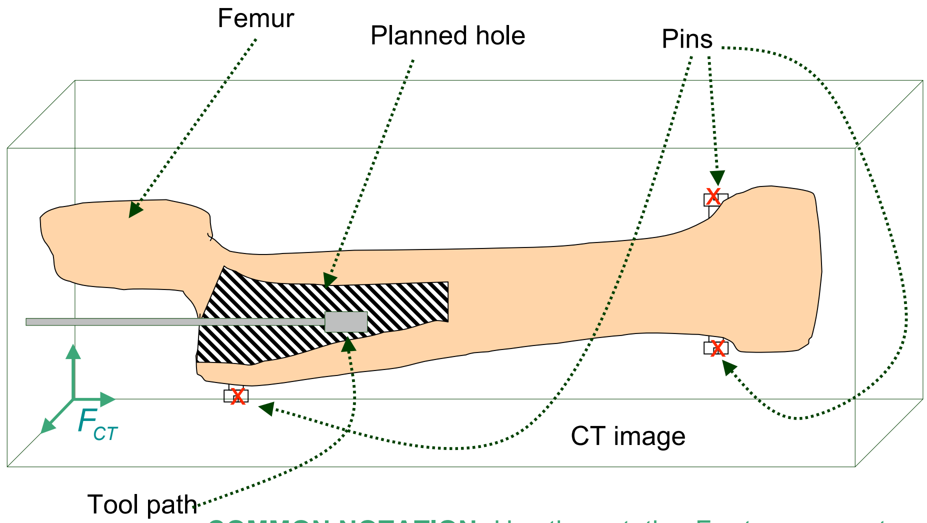
$$\vec{p}_{cutter} = \vec{p}_{hole} + \vec{p}_{hc}$$

Defining things relative to other things

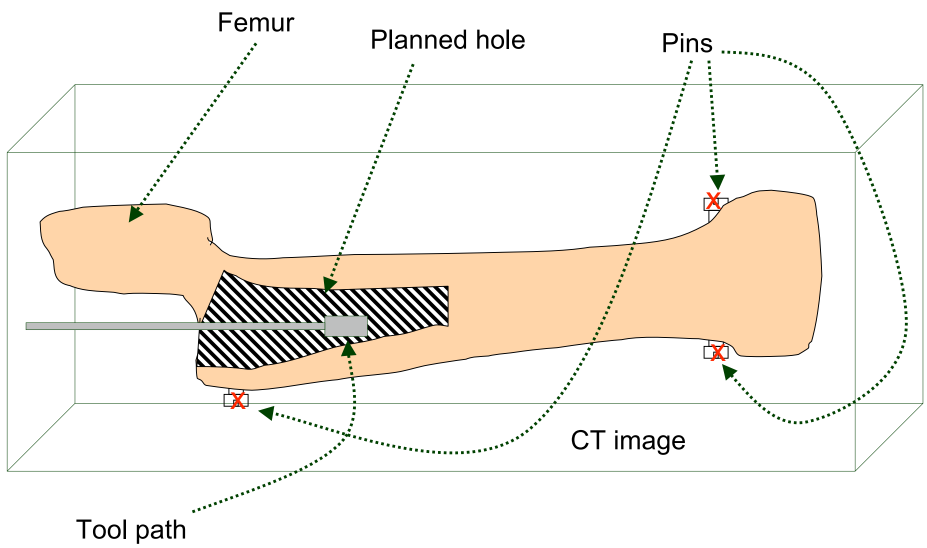


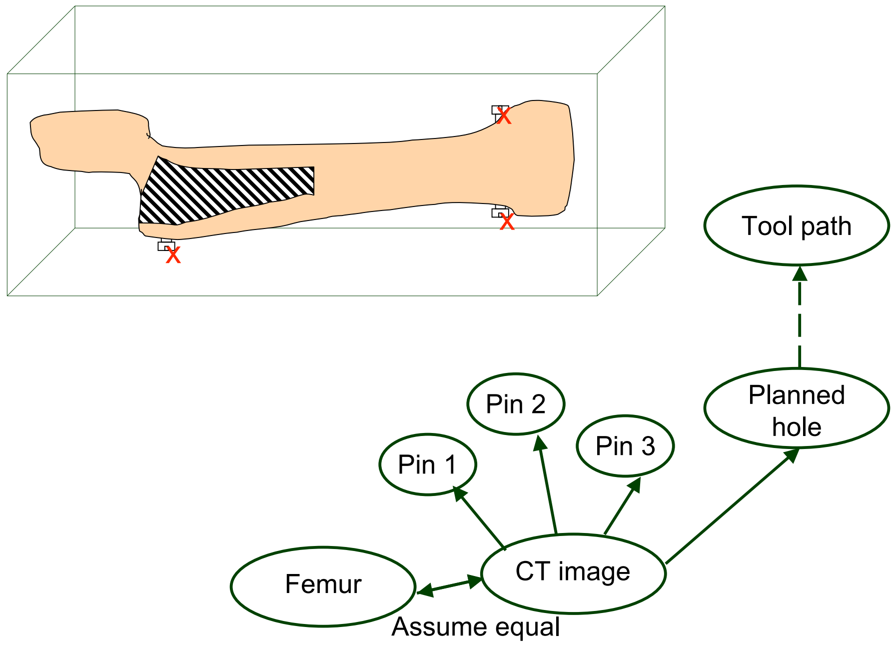


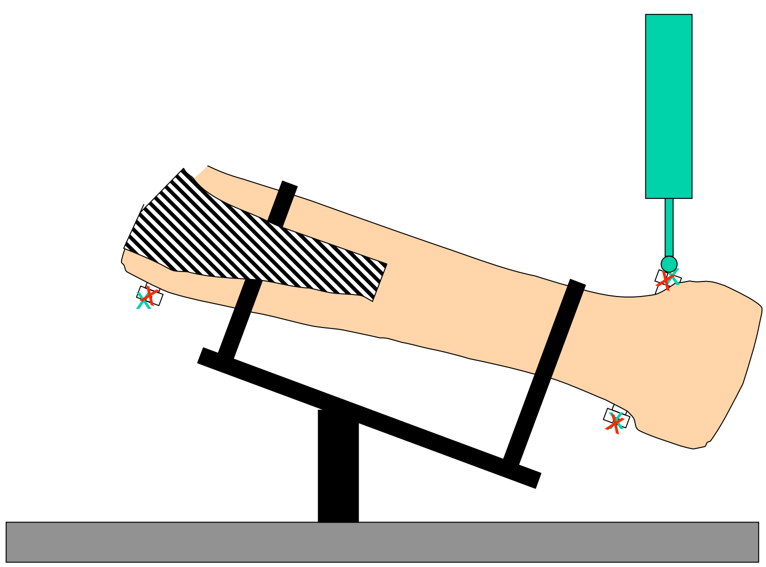
$$\vec{p}_{cutter} = \vec{p}_{pin} - \vec{p}_{bp} + \vec{p}_{bh} + \vec{p}_{hc}$$

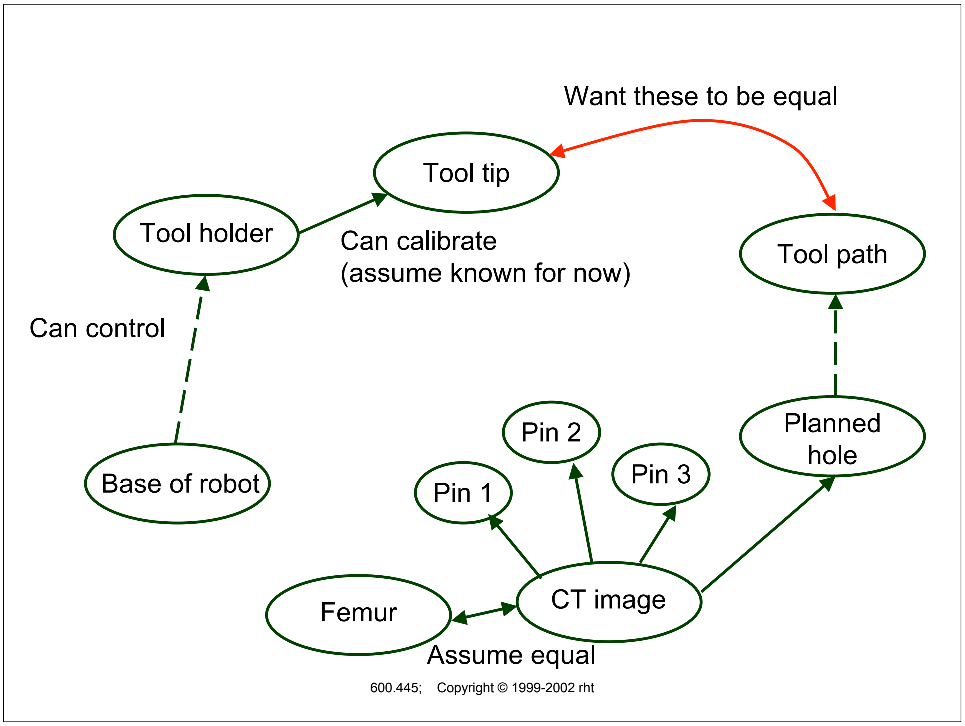


COMMON NOTATION: Use the notation F_{obj} to represent a coordinate system or the position and orientation of an object (relative to some unspecified coordinate system). Use $F_{x,y}$ to mean position and orientation of y relative to x.




Tool path
Planned
hole
Pin 2
Pin 3
Pin 1
CT image
Femur
Assume equal




Want these to be equal
Tool tip
Tool holder
Tool path
Can calibrate
(assume known for now)
Can control
Pin 2
Pin 3
Planned hole
Base of robot
Pin 1
Femur
CT image
Assume equal
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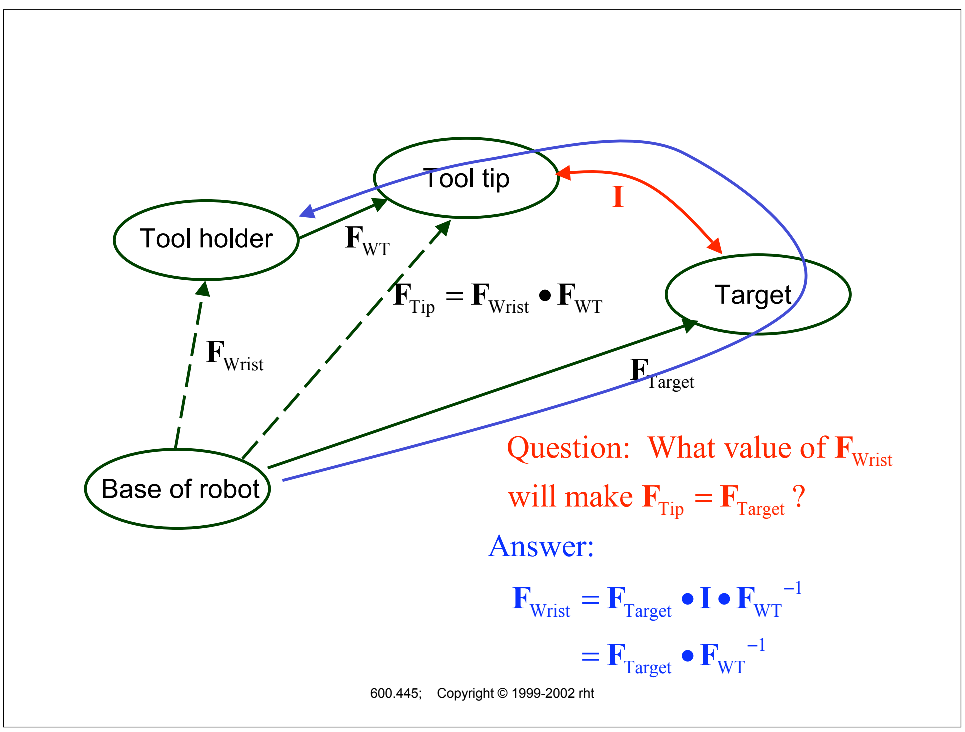
Tool tip
I
Tool holder
F_{WT}
$F_{Tip} = F_{Wrist} \bullet F_{WT}$
Target
F_{Wrist}
F_{Target}
Question: What value of F_{Wrist}
will make $F_{Tip} = F_{Target}$?
Base of robot
Answer:
$F_{Wrist} = F_{Target} \bullet I \bullet F_{WT}^{-1}$
$= F_{Target} \bullet F_{WT}^{-1}$
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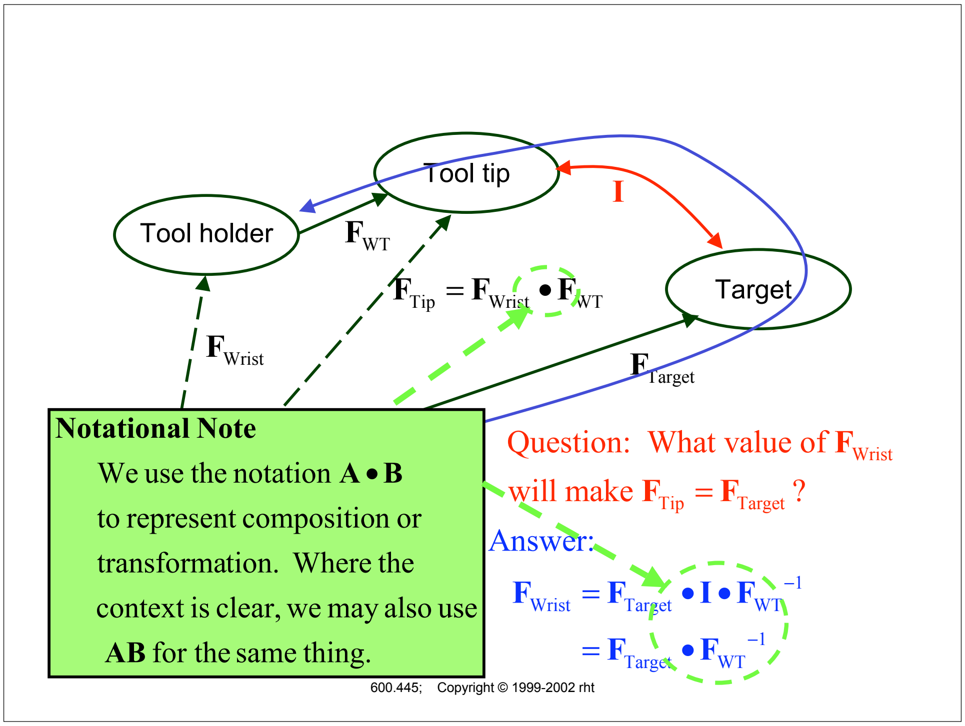
Tool tip
Tool holder
Target
I
$\mathbf{F}_{WT}$
$\mathbf{F}_{Tip} = \mathbf{F}_{Wrist} \bullet \mathbf{F}_{WT}$
$\mathbf{F}_{Wrist}$
$\mathbf{F}_{Target}$
Notational Note
We use the notation $\mathbf{A} \bullet \mathbf{B}$ to represent composition or transformation. Where the context is clear, we may also use $\mathbf{AB}$ for the same thing.
Question: What value of $\mathbf{F}_{Wrist}$ will make $\mathbf{F}_{Tip} = \mathbf{F}_{Target}$?
Answer:
$\mathbf{F}_{Wrist} = \mathbf{F}_{Target} \bullet \mathbf{I} \bullet \mathbf{F}_{WT}^{-1}$
$= \mathbf{F}_{Target} \bullet \mathbf{F}_{WT}^{-1}$
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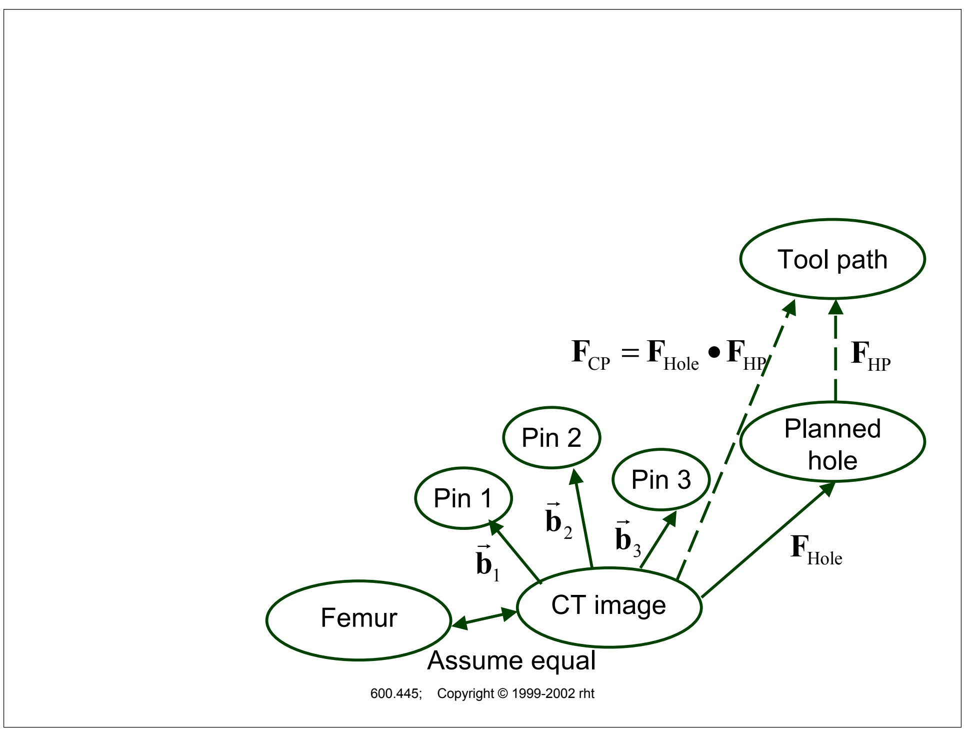
Tool path
$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$
$\mathbf{F}_{HP}$
Planned hole
Pin 2
Pin 3
Pin 1
$\vec{\mathbf{b}}_2$
$\vec{\mathbf{b}}_3$
$\vec{\mathbf{b}}_1$
$\mathbf{F}_{Hole}$
Femur
CT image
Assume equal
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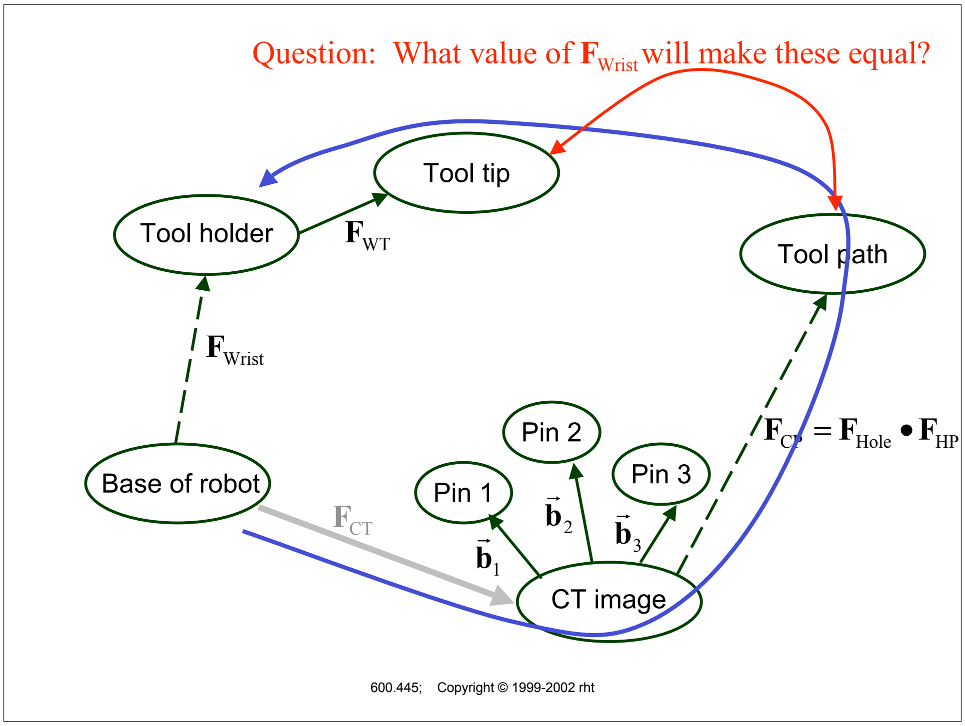
Question: What value of $\mathbf{F}_{Wrist}$ will make these equal?
Tool tip
Tool holder
$\mathbf{F}_{WT}$
Tool path
$\mathbf{F}_{Wrist}$
$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$
Pin 2
Pin 3
Pin 1
Base of robot
$\mathbf{F}_{CT}$
$\vec{\mathbf{b}}_2$
$\vec{\mathbf{b}}_3$
$\vec{\mathbf{b}}_1$
CT image
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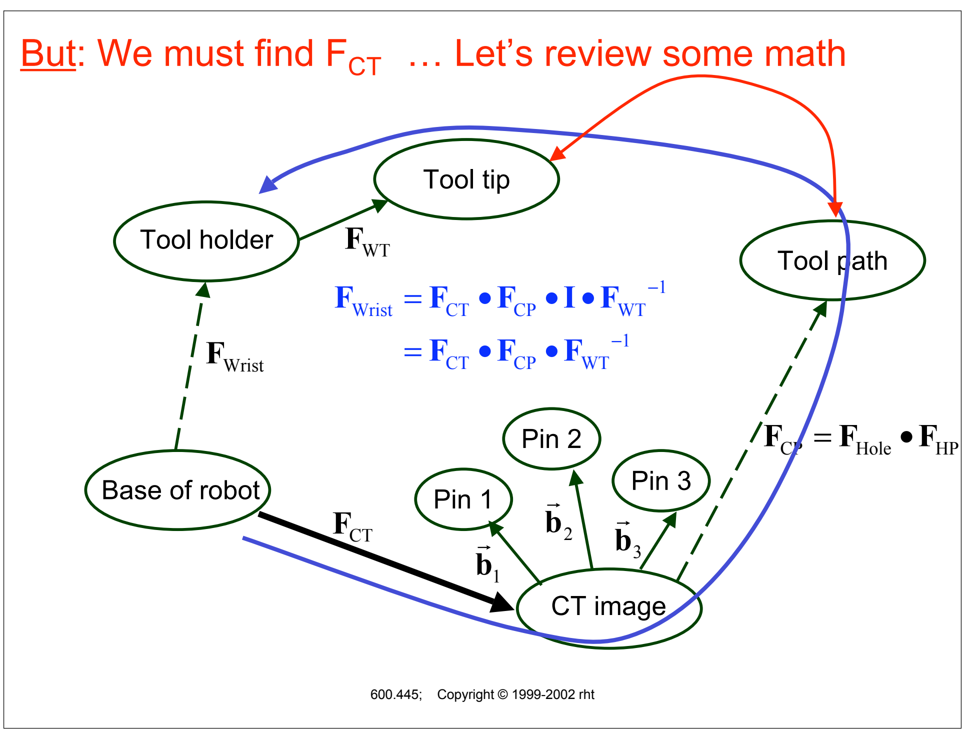
But: We must find F_{CT} … Let's review some math
Tool tip
Tool holder
$\mathbf{F}_{WT}$
Tool path
$\mathbf{F}_{Wrist} = \mathbf{F}_{CT} \bullet \mathbf{F}_{CP} \bullet \mathbf{I} \bullet \mathbf{F}_{WT}^{-1}$
$= \mathbf{F}_{CT} \bullet \mathbf{F}_{CP} \bullet \mathbf{F}_{WT}^{-1}$
$\mathbf{F}_{Wrist}$
$\mathbf{F}_{CP} = \mathbf{F}_{Hole} \bullet \mathbf{F}_{HP}$
Pin 2
Pin 3
Pin 1
Base of robot
$\mathbf{F}_{CT}$
$\vec{\mathbf{b}}_2$
$\vec{\mathbf{b}}_3$
$\vec{\mathbf{b}}_1$
CT image
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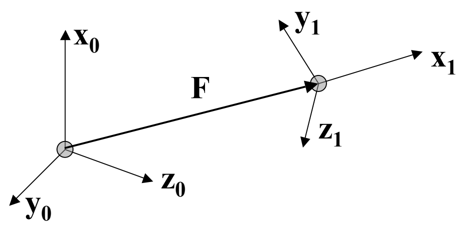
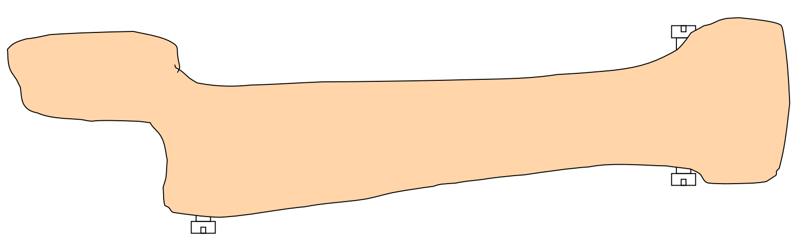
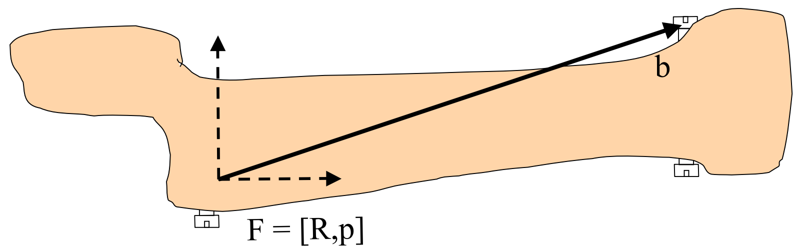
Coordinate Frame Transformation

$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$



Slide acknowledgment: Sarah Graham and Andy Bzostek





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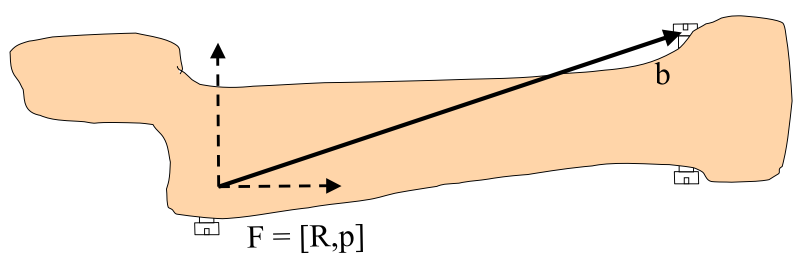

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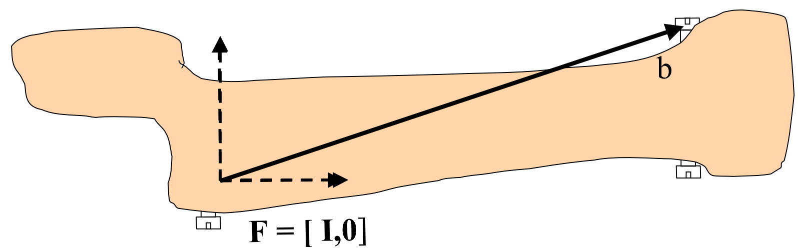
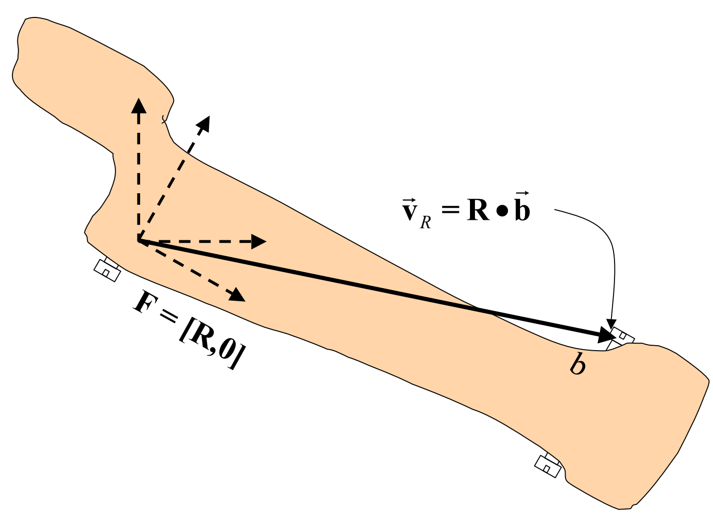
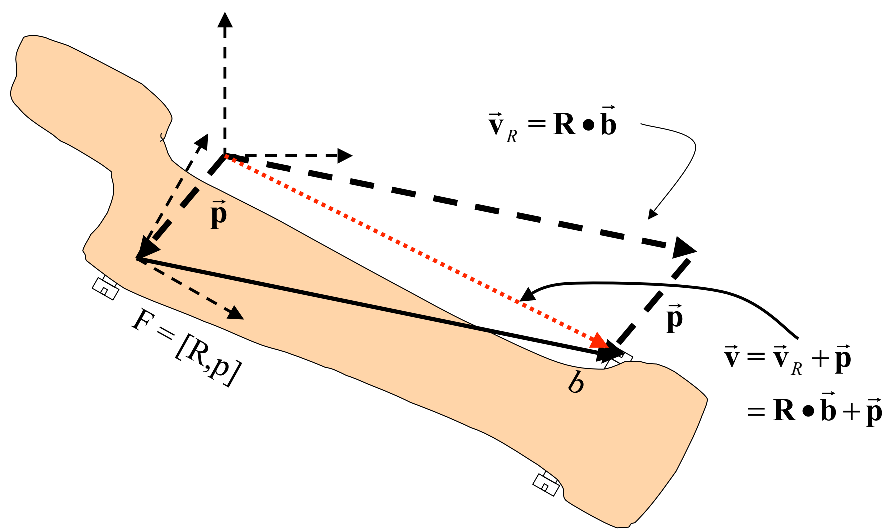


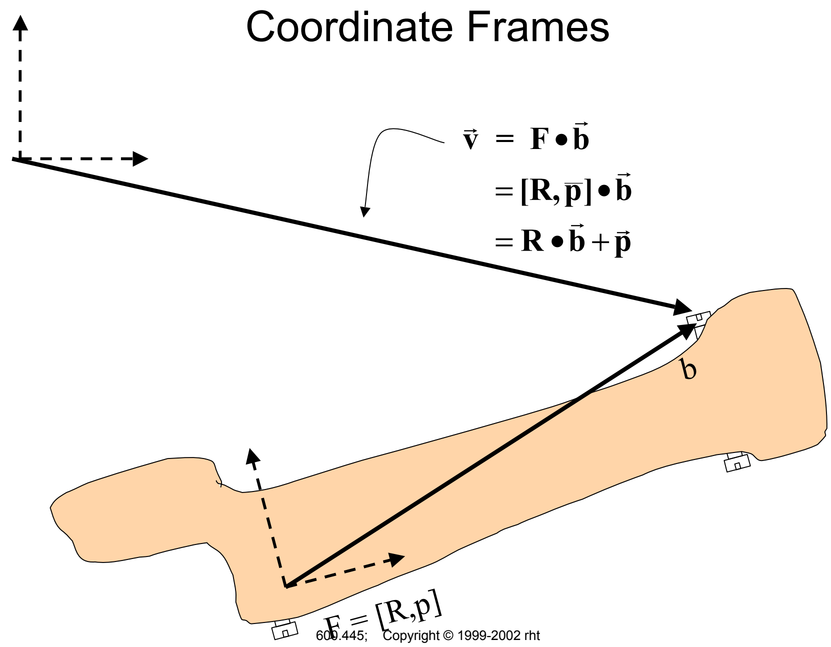
b
F = [I,0]



$\vec{v}_R = R \bullet \vec{b}$
b
F = [R,0]



Coordinate Frames



Forward and Inverse Frame Transformations

Forward

$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$

$$\begin{aligned}\mathbf{v} &= \mathbf{F} \bullet \mathbf{b} \\ &= [\mathbf{R}, \mathbf{p}] \bullet \mathbf{b} \\ &= \mathbf{R} \bullet \mathbf{b} + \mathbf{p}\end{aligned}$$

Inverse

$$\mathbf{F}^{-1}\mathbf{v} = \mathbf{b}$$

$$\begin{aligned}\mathbf{b} &= \mathbf{R}^{-1} \bullet (\mathbf{v} - \mathbf{p}) \\ &= \mathbf{R}^{-1} \bullet \mathbf{v} - \mathbf{R}^{-1} \bullet \mathbf{p}\end{aligned}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \mathbf{p}]$$

Composition

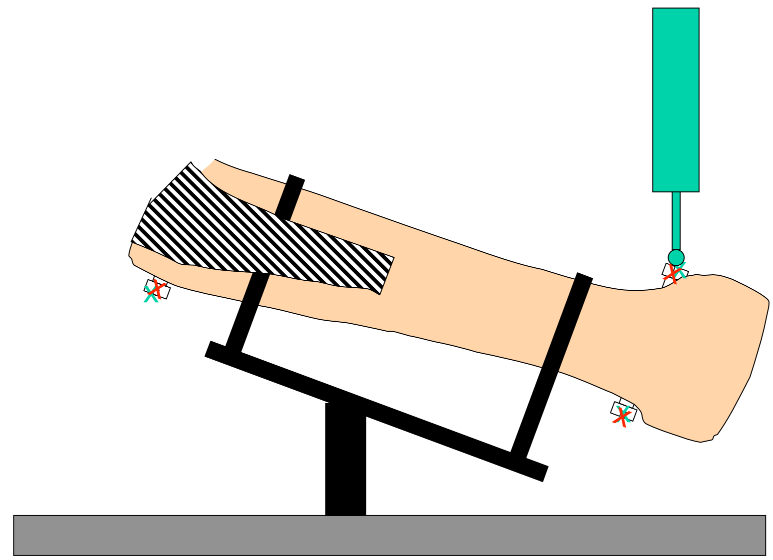
Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1], \quad \mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

Then

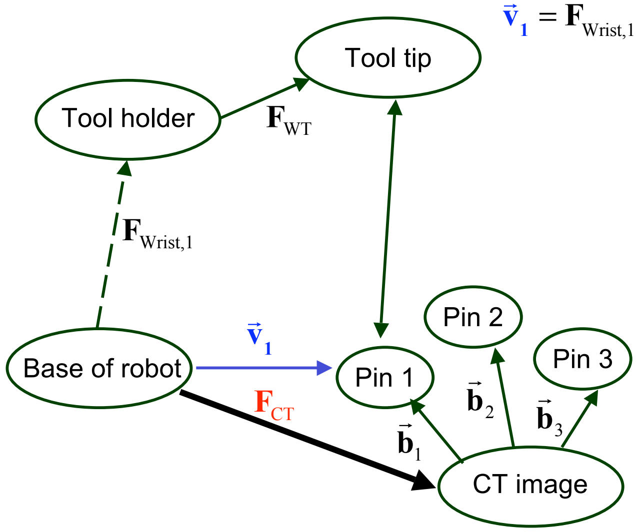
$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\ &= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\ &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\ &= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}\end{aligned}$$

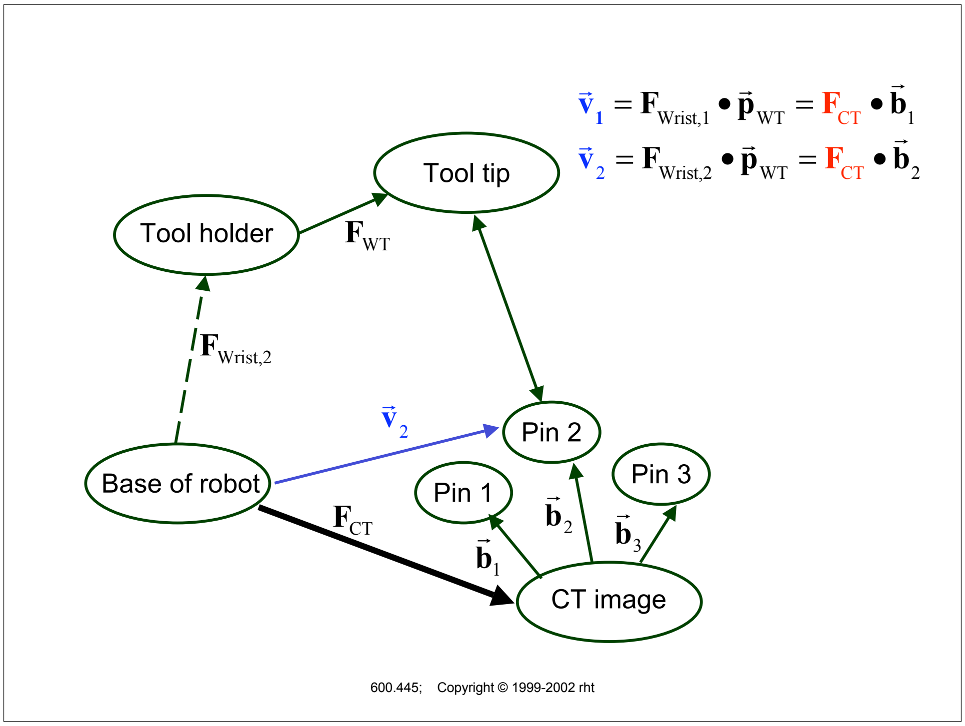
So

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]\end{aligned}$$



$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\text{Wrist},1} \bullet \vec{\mathbf{p}}_{\text{WT}} = \mathbf{F}_{\text{CT}} \bullet \vec{\mathbf{b}}_1$$




$\vec{v}_1 = F_{Wrist,1} \bullet \vec{p}_{WT} = F_{CT} \bullet \vec{b}_1$
$\vec{v}_2 = F_{Wrist,2} \bullet \vec{p}_{WT} = F_{CT} \bullet \vec{b}_2$
Tool tip
Tool holder
F_{WT}
$F_{Wrist,2}$
$\vec{v}_2$
Pin 2
Pin 3
Pin 1
Base of robot
F_{CT}
$\vec{b}_1$
$\vec{b}_2$
$\vec{b}_3$
CT image
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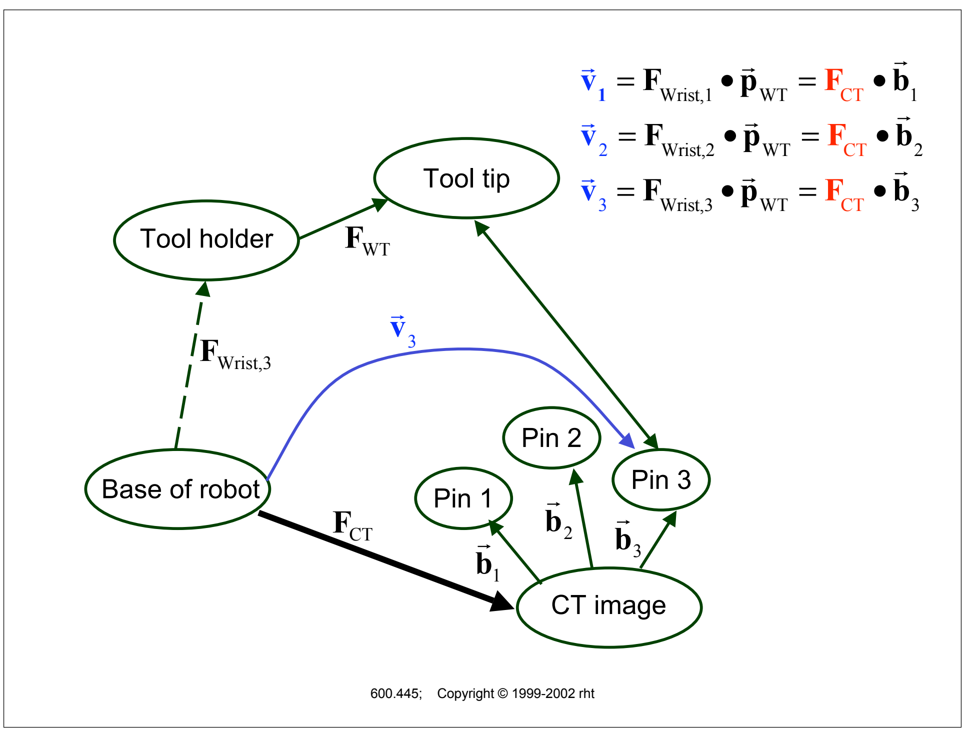
$\vec{v}_1 = F_{Wrist,1} \bullet \vec{p}_{WT} = F_{CT} \bullet \vec{b}_1$
$\vec{v}_2 = F_{Wrist,2} \bullet \vec{p}_{WT} = F_{CT} \bullet \vec{b}_2$
$\vec{v}_3 = F_{Wrist,3} \bullet \vec{p}_{WT} = F_{CT} \bullet \vec{b}_3$
Tool tip
Tool holder
F_{WT}
$F_{Wrist,3}$
$\vec{v}_3$
Pin 2
Pin 3
Pin 1
Base of robot
F_{CT}
$\vec{b}_1$
$\vec{b}_2$
$\vec{b}_3$
CT image
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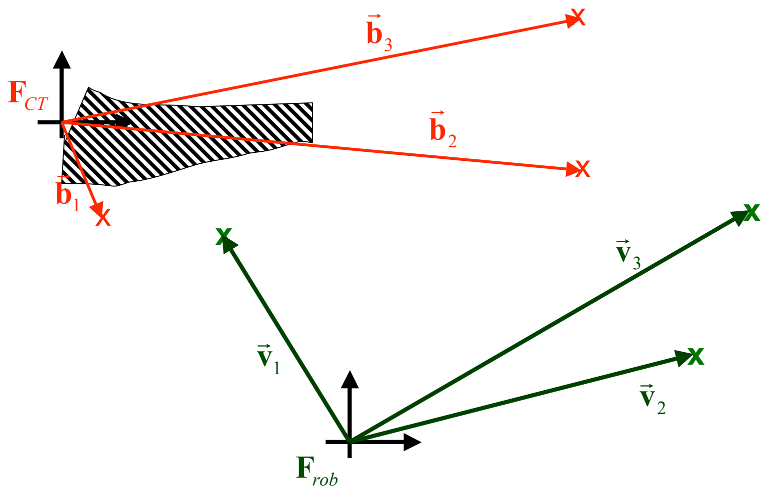
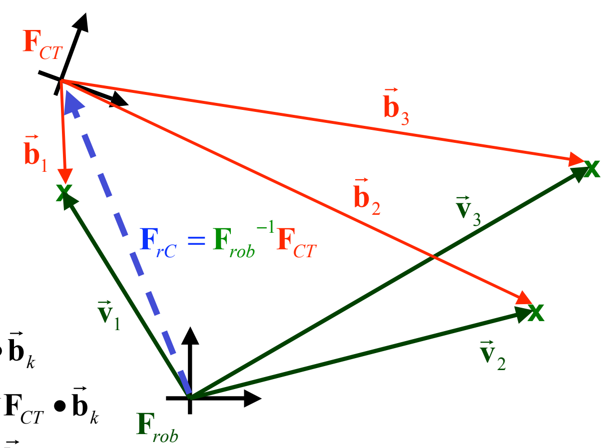
Frame transformation from 3 point pairs



Frame transformation from 3 point pairs



$$F_{rob} \bullet \vec{v}_k = F_{CT} \bullet \vec{b}_k$$

$$\vec{v}_k = F_{rob}{}^{-1} F_{CT} \bullet \vec{b}_k$$

$$\vec{v}_k = F_{rC} \bullet \vec{b}_k$$

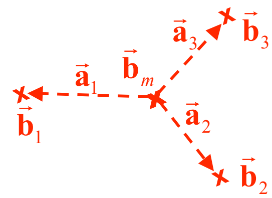
Frame transformation from 3 point pairs

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC}\vec{\mathbf{b}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC}$$

Define

$$\vec{\mathbf{v}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{v}}_k \qquad \vec{\mathbf{b}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{b}}_k$$

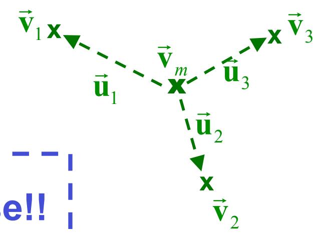
$$\vec{\mathbf{u}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m \qquad \vec{\mathbf{a}}_k = \vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m$$



$$\mathbf{F}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC} = \mathbf{R}_{rC}\left(\vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m\right) + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC} - \mathbf{R}_{rC}\vec{\mathbf{b}}_m - \vec{\mathbf{p}}_{rC}$$



$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m = \vec{\mathbf{u}}_k$$

$$\vec{\mathbf{p}}_{rC} = \vec{\mathbf{u}}_m - \mathbf{R}_{rC}\vec{\mathbf{b}}_m$$

Solve These!!

Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{\mathbf{a}}_k = \vec{\mathbf{u}}_k$ for $k = 1,\cdots,n$ the problem is to estimate $\mathbf{R}$. This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = [\vec{\mathbf{u}}_1 \ \cdots \ \vec{\mathbf{u}}_n]$ and $\mathbf{A} = [\vec{\mathbf{a}}_1 \ \cdots \ \vec{\mathbf{a}}_n]$

Step 2: Solve the system $\mathbf{RA} = \mathbf{U}$ for $\mathbf{R}$. E.g., by $\mathbf{R} = \mathbf{UA}^{-1}$

Step 3: Renormalize $\mathbf{R}$ to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = \left[\vec{\mathbf{r}}_x \mid \vec{\mathbf{r}}_y \mid \vec{\mathbf{r}}_z \right]$, modify it so $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

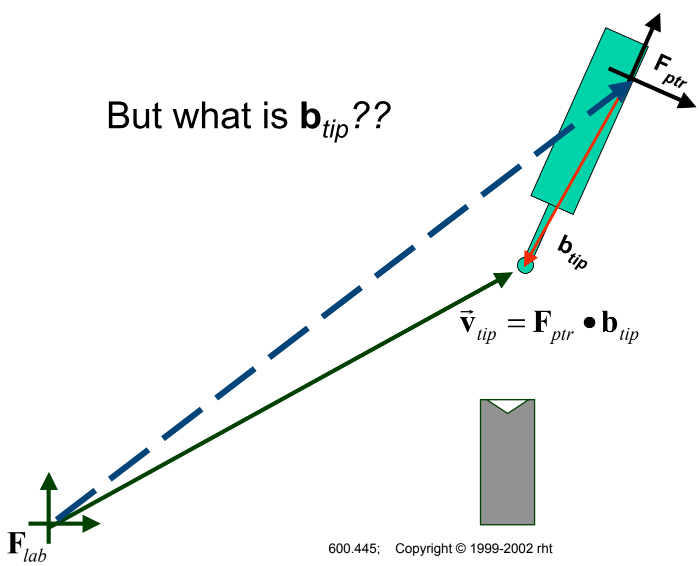
Step 1: $\vec{\mathbf{a}} = \vec{\mathbf{r}}_y \times \vec{\mathbf{r}}_z$

Step 2: $\vec{\mathbf{b}} = \vec{\mathbf{r}}_z \times \vec{\mathbf{a}}$

Step 3: $\mathbf{R}_{normalized} = \left[\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \middle| \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \middle| \frac{\vec{\mathbf{r}}_z}{\|\vec{\mathbf{r}}_z\|} \right]$

Calibrating a pointer

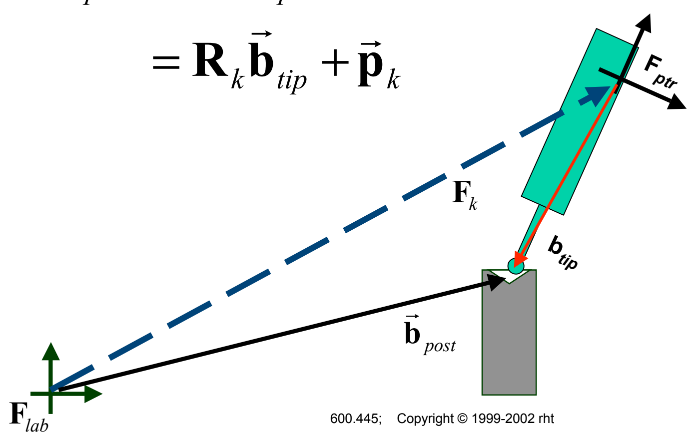
But what is $\mathbf{b}_{tip}$??



$\vec{\mathbf{v}}_{tip} = \mathbf{F}_{ptr} \bullet \mathbf{b}_{tip}$

Calibrating a pointer

$$\vec{\mathbf{b}}_{post} = \mathbf{F}_k \vec{\mathbf{b}}_{tip}$$
$$= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$



Calibrating a pointer

For each measurement k, we have

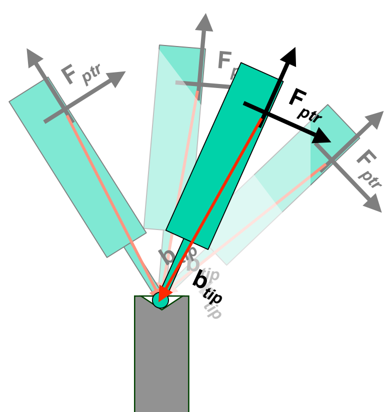
$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

I. e.,

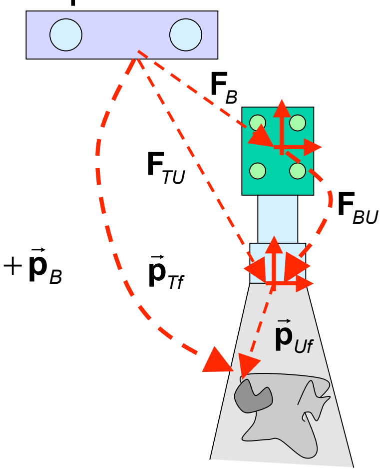
$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\begin{bmatrix} \vdots & \vdots \\ \mathbf{R}_k & -\mathbf{I} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{tip} \\ \vec{\mathbf{b}}_{post} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{bmatrix}$$



Another Example



$$\vec{\mathbf{p}}_{Tf} = \mathbf{F}_{TU} \bullet \vec{\mathbf{p}}_{Uf}$$

$$\begin{aligned}\mathbf{F}_{TU} &= \mathbf{F}_B \bullet \mathbf{F}_{BU} \\ &= [\mathbf{R}_B \bullet \mathbf{R}_{BU}, \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B]\end{aligned}$$

$$\vec{\mathbf{p}}_{Tf} = \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B$$

Also

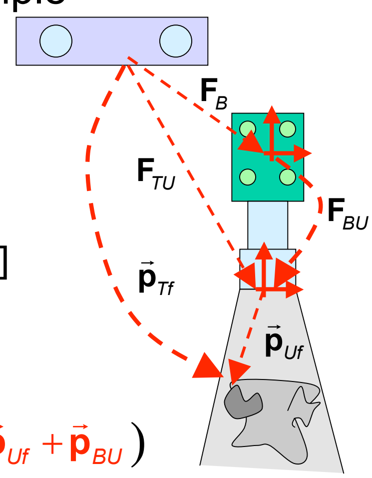
$$\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \bullet \vec{\mathbf{p}}_{Bf}$$

$$\begin{aligned}\vec{\mathbf{p}}_{Bf} &= \mathbf{F}_{BU} \bullet \vec{\mathbf{p}}_{Uf} \\ &= \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU}\end{aligned}$$

$$\vec{\mathbf{p}}_{Tf} = \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B$$

Another Example

Suppose that the track body to US calibration is not perfect



$$\begin{aligned}\mathbf{F}_{BU}^{*} &= \mathbf{F}_{BU} \Delta \mathbf{F}_{BU} \\ &= [\mathbf{R}_{BU} \Delta \mathbf{R}_{BU}, \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_{BU}]\end{aligned}$$

$$\begin{aligned}\Delta \vec{\mathbf{p}}_{Bf} &= \mathbf{F}_{BU} \Delta \mathbf{F}_{BU} \vec{\mathbf{p}}_{Uf} - \vec{\mathbf{p}}_{Bf} \\ &= \mathbf{F}_{BU} (\Delta \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} + \Delta \vec{\mathbf{p}}_{BU}) - (\mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU}) \\ &= \mathbf{R}_{BU} \Delta \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} - \vec{\mathbf{p}}_{BU} \\ &= \mathbf{R}_{BU} \Delta \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \Delta \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU} \vec{\mathbf{p}}_{Uf}\end{aligned}$$

Another Example

Continuing …

$$
\begin{aligned}
\Delta\vec{\mathbf{p}}_{Bf} &= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} \\
&\approx \mathbf{R}_{BU}\left(\mathbf{I} + \mathbf{skew}(\vec{\alpha}_{BU})\right)\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} \\
&= \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} \\
&= \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} \\
&= -\mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} \times \vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} \\
&= \mathbf{R}_{BU}\,skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}
\end{aligned}
$$

Another Example

$$
\vec{\mathbf{p}}_{Tf} + \Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B\Delta\mathbf{F}_B\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right)
$$

$$
\Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B\left(\Delta\mathbf{F}_B\vec{\mathbf{p}}_{Bf} + \Delta\mathbf{F}_B\Delta\vec{\mathbf{p}}_{Bf}\right) - \mathbf{F}_B\vec{\mathbf{p}}_{Bf}
$$

$$
\begin{aligned}
\Delta\mathbf{F}_B\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) &= \Delta\mathbf{R}_B\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B \\
&\approx \left(\mathbf{I} + skew(\vec{\alpha}_B)\right)\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B \\
&= \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \Delta\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B \\
&\approx \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B
\end{aligned}
$$

$$
\begin{aligned}
\Delta\vec{\mathbf{p}}_{Tf} &\approx \mathbf{F}_B\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) - \mathbf{F}_B\vec{\mathbf{p}}_{Bf} \\
&= \mathbf{R}_B\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) + \vec{\mathbf{p}}_B - \left(\mathbf{R}_B\vec{\mathbf{p}}_{Bf} + \vec{\mathbf{p}}_B\right) \\
&= \mathbf{R}_B\left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)
\end{aligned}
$$

$$
\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU}\,skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}
$$

Another Example

$$\vec{\mathbf{p}}_{Tf} + \Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right)$$

$$\Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) - \mathbf{F}_B\vec{\mathbf{p}}_{Bf}$$

$$\begin{aligned}
\Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) &= \Delta\mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B \\
&\approx \left(\mathbf{I} + skew(\vec{\alpha}_B)\right)\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B \\
&= \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \Delta\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B \\
&\approx \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B
\end{aligned}$$

$$\begin{aligned}
\Delta\vec{\mathbf{p}}_{Tf} &\approx \mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) - \mathbf{F}_B\vec{\mathbf{p}}_{Bf} \\
&= \mathbf{F}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)
\end{aligned}$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$