Modern Complexity Theory  
Spring 2004  
Midterm Exam

Problem 1: Completeness  (7 points)

(a) Consider any $C$-complete language $L$ for some complexity class $C$. Show that for every language $L' \in C$ with $L \leq_p L'$ it holds that also $L'$ is $C$-complete. (2 points)

(b) Recall that $QSAT = \{ \langle \psi \rangle \mid \psi \text{ is a valid quantified Boolean expression in conjunctive prenex normal form} \}$. Consider now the decision problem

$$GAME = \{ \langle \psi \rangle \mid \psi \text{ is a valid quantified Boolean expression of the form } \exists x_1 \forall x_2 \exists x_3 \ldots Q_n x_n \phi \text{ for some } n \in \mathbb{N}, \text{ where } \phi \text{ is an expression in CNF with variables } x_1, \ldots, x_n \}$$

Show that $GAME$ is PSPACE-complete. (2 points)

(c) Show that PARITH restricted to variables with values in the range $\{0, \ldots, k\}$ for some constant $k \geq 1$ is PSPACE-complete. (3 points)

Problem 2: Undecidability  (3 points)

Use the fact that the language

$$L_d = \{ \langle M \rangle \mid M \text{ started with } \langle M \rangle \text{ does not accept} \}$$

is undecidable to show that also the language

$$E = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^* \}$$

is undecidable.

Problem 3: Set Theory, Logic, and Arithmetic  (6 points)

(a) Show that $|\mathbb{R}| = |\mathbb{R}^2|$.

(b) Show that for all quantified Boolean expressions $\phi_1$ and $\phi_2$ it holds that

$$\forall x (\phi_1 \land \phi_2) \equiv (\forall x \phi_1 \land \forall x \phi_2).$$

(c) Give an elementary arithmetical expression for the statement “$x$ is the greatest common divisor of $a$ and $b$”.

Good luck!