CONVEX REPRESENTATIONS OF GRAPHS

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1. Introduction

A *graph* G is defined by a set $V(G)$ of *vertices*, a set $E(G)$ of *edges*, and a relation of *incidence* which associates with each edge two vertices, not necessarily distinct, called its *ends*. If the two ends of an edge coincide the edge is a *loop*; otherwise it is a *link*.

In this paper we consider only *finite* graphs, that is graphs for which $V(G)$ and $E(G)$ are both finite. We denote the numbers of edges and vertices by $\alpha_1(G)$ and $\alpha_0(G)$ respectively.

A graph is *simple* if it has no loop, and no pair of edges with the same two ends.

Let G be a simple graph, and let f be a 1–1 mapping of $V(G)$ onto a set S of $\alpha_0(G)$ distinct points in the euclidean plane E^2. If e is an edge of G with ends v and w we write $f(e)$ for the straight segment joining $f(v)$ and $f(w)$. We can define a graph H by postulating that $V(H) = S$, $E(H)$ is the set of all segments $f(e)$, and the incident vertices of an edge $f(e)$ of H are its two ends in the geometric sense $(e \in E(G))$. We call H a *straight representation* of G in E^2 if it satisfies the following two conditions.

(i) If an edge $f(e)$ of H contains a point $f(v)$ of S, then $f(v)$ is an end of $f(e)$.

(ii) If two distinct edges of H have a common point x, then x is an end of each of them.

A straight representation H of G may separate the rest of the plane into a finite number of regions, each of which is the interior or exterior of a convex polygon. We then call H a *convex representation* of G in E^2. Our object in this paper is to establish necessary and sufficient conditions for the existence of a convex representation of a given simple graph G. We try to do this using only methods of elementary geometry and combinatorial graph theory. Accordingly we base our treatment not on the usual theory of 'planar graphs', but on the 'planar meshes' of graphs defined combinatorially below. In the remainder of this introductory section we give some of the definitions to be used and then state the main result of the paper.

Let G be any graph. We define the *degree* $d(G,v)$ of a vertex v as the number of incident edges, loops being counted twice. It is clear that

$$(1.1) \qquad \sum_{v \in V(G)} d(G,v) = 2\alpha_1(G).$$

If $d(G,v) = 0$ we call v an *isolated* vertex of G.

A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and each edge of H has the same ends in H as in G. We then write $H \subseteq G$. If in addition H is not identical with G we call H a *proper* subgraph of G and write $H \subset G$. If $V(H)$ and $E(H)$ are both null then H is the *null* subgraph of G, denoted in formulae by the symbol $\emptyset$. If $S \subseteq E(G)$ we write $G \cdot S$ for the subgraph of G defined by the members of S and their incident vertices.

A *path* in G is a finite sequence

$$P = (v_0, e_1, v_1, e_2,..., e_k, v_k),$$

having at least one term, which satisfies the following conditions.

(i) The terms of P are alternately vertices v_i and edges e_j of G.

(ii) If $0 < i \leqslant k$, then v_{i-1} and v_i are the two ends of e_i in G.

The path P is *degenerate* if it has only the term v_0. It is *simple* if all its terms are distinct. It is *circular* if it is non-degenerate and all its terms are distinct except that $v_0 = v_k$. It is easily seen that P always has a subsequence which is a simple path in G from v_0 to v_k.

The edges and vertices of a path P define a subgraph $G(P)$ of G. We call $G(P)$ an *arc-graph* or *circuit* of G if P is a non-degenerate simple path or a circular path respectively. In the first case we call v_0 and v_k the *ends* of the arc-graph and the other vertices, if any, the *internal* vertices.

Let U_1, U_2,..., U_k be subsets of a given set U, not necessarily all distinct. Their *mod 2 sum* is the set of all $u \in U$ such that the number of suffixes i satisfying $u \in U_i$ is odd. It is easily seen that this addition of subsets is commutative and associative. We use the ordinary additive notation to represent it.

A *cycle* of a graph G is a subset K of $E(G)$ such that the number of links of G in K incident with any vertex of G is even. Clearly any mod 2 sum of cycles of G is itself a cycle of G. If C is any circuit of G then $E(C)$ is clearly a cycle. We call a cycle of this kind *elementary*.

A *planar mesh* of G is a set $\mathbf{M} = \{S_1, S_2,..., S_k\}$ of elementary cycles of G, not necessarily all distinct, which satisfies the following conditions.

(i) If an edge of G belongs to one of the sets S_i it belongs to just two of them.

(ii) Each non-null cycle of G can be expressed as a mod 2 sum of members of $\mathbf{M}$.†

Condition (i) implies that no elementary cycle of G can appear more than twice as a member of $\mathbf{M}$. Further, the mod 2 sum of all the members of $\mathbf{M}$ is null.

A graph G is *separable* if it has two proper subgraphs H and K, with no

† See S. MacLane, 'A combinatorial condition for planar graphs', *Fundam. Math.* 28 (1937) 22–32.

common edge and at most one common vertex, which together include all the edges and vertices of G. Section 3 of this paper is concerned mainly with the planar meshes of non-separable graphs. A planar mesh $\mathbf{M}$ of such a graph is shown to have the property of *connexion*, that is no non-null proper subset of $\mathbf{M}$ has a null mod 2 sum.

Let C and C' be disjoint elementary cycles of a graph G. We denote by $\lambda(C, C')$ the least integer n such that there are complementary subsets S and S' of $E(G)$ satisfying the following conditions.

(i) $C \subseteq S$ and $C' \subseteq S'$.

(ii) The number of vertices incident with members of both sets S and S' is n.

We say G is 3-*linked* with respect to an elementary cycle C if there is no elementary cycle C' of G such that $C \cap C'$ is null and $\lambda(C, C') \leqslant 2$.

In § 4 the terms *node* and *branch* are defined, a node being a vertex whose degree is not 2. We note here that a 'branch of G spanning the circuit H' is an arc-graph having both ends, but no edge or internal vertex, in H, and whose internal vertices are not incident with any edges of G other than those of the arc-graph.

Consider a convex representation R of a simple graph G in a plane Π, R being determined by a 1–1 mapping f of $V(G)$ onto a set of points of Π.

R separates the remainder of Π into a finite number of regions $Q_1, Q_2, \ldots, Q_k$, each of which is the interior or exterior of a convex polygon. One of these regions, Q_1 say, is of infinite area. It must be the exterior of a convex polygon P_1. The other regions Q_i are all contained in the interior of P_1. So if $2 \leqslant i \leqslant k$ then Q_i is the interior of a convex polygon P_i. Accordingly R determines a dissection of the convex polygon P_1 into convex polygons $P_2, \ldots, P_k$.

We call R *singular* if there is a node x of G and a polygon P_i $(1 \leqslant i \leqslant k)$ such that $f(x)$ is in the boundary of P_i but is not a vertex of P_i. In other words the region Q_i occupies an angle of exactly two right angles at $f(x)$. We then call x a *singular node* of G with respect to R.

The boundary of P_i is a union of edges of R. These edges make up an elementary cycle of R, the image under f of an elementary cycle C_i of G $(1 \leqslant i \leqslant k)$. We call $C_1, C_2, \ldots, C_k$ the *outer cycles*, and C_1 the *frame*, of G with respect to f or R.

The main theorems of the paper are as follows.

THEOREM I. *Let a simple graph G have outer cycles $C_1, \ldots, C_k$, and frame C_1, with respect to a convex representation R. Then $\{C_1, \ldots, C_k\}$ is a connected planar mesh of G, G is non-separable and* 3-*linked with respect to C_1, and no branch of G spanning $G \cdot C_1$ has the same ends as some edge of C_1.*

THEOREM II. *Let* **M** *be a planar mesh* $\{C_1,..., C_k\}$ *of a simple graph* G. *Let* G *be non-separable and* 3-*linked with respect to* C_1, *and let no branch of* G *spanning* $G \cdot C_1$ *have the same two ends as any edge of* C_1.

Let P_1 *be any* $\alpha_0(G \cdot C_1)$-*sided convex polygon in a plane* Π, *and let* h *be a* 1–1 *mapping of* $V(G \cdot C_1)$ *onto the set of vertices of* P_1 *which preserves the cyclic order.*

Then we can find a 1–1 *mapping* f *of* $V(G)$ *onto a set of points of* Π *such that the following conditions are satisfied*:

(i) $$f(x) = h(x) \quad if \quad x \in V(G \cdot C_1),$$

(ii) f *determines a non-singular convex representation* R *of* G, *with respect to which* G *has the frame* C_1 *and the outer cycles* $C_1,..., C_k$.

2. If $H_1, H_2,..., H_k$ are subgraphs of G we define their *union*

$$H_1 \cup H_2 \cup ... \cup H_k$$

(*intersection* $H_1 \cap H_2 \cap ... \cap H_k$) as the subgraph H of G such that $V(H)$ is the union (intersection) of the sets $V(H_i)$ and $E(H)$ is the union (intersection) of the sets $E(H_i)$. We say the subgraphs $H_1, H_2,..., H_k$ are *disjoint* if

$$H_i \cap H_j = \emptyset$$

whenever $0 \leqslant i < j \leqslant k$.

We say that a path P *avoids* a subgraph H of G if no term of P other than the first and the last is an edge or vertex of H.

Let A_1, A_2, and A_3 be three arc-graphs of G which have a common end v, and no two of which have any other vertex, or any edge, in common. Then we call their union a Y-*graph* of G. The three arc-graphs are the *arms* of the Y-graph, their ends other than v are its *ends*, and v is its *centre*.

We find it convenient to say that an arc-graph or Y-graph L *spans* a subgraph H if $E(H) \cap E(L)$ is null and $V(H) \cap V(L)$ consists solely of the ends of L.

A *vertex of attachment* of a subgraph H of G is a vertex of H which is incident, as a vertex of G, with an edge of $E(G)-E(H)$.

Let J be a fixed subgraph of G. We say a subgraph H of G is J-*bounded* if all its vertices of attachment belong to J. Clearly any union or intersection of J-bounded subgraphs of G is itself J-bounded.

Let **J** be the class of all subgraphs H of G such that H is J-bounded and not a subgraph of J. A *bridge* of J in G is any member of **J** which has no other member of **J** as a subgraph.

(2.1) *Let* B *be a bridge of* J *in* G. *Then no edge or isolated vertex of* B *belongs to* J.

Proof. Write $E = E(B) \cap E(J)$ and let V denote the set of all isolated vertices of B belonging to J. There is a subgraph B' of B such that

$$E(B') = E(B) - E \quad \text{and} \quad V(B') = V(B) - V.$$

Now B' is not a subgraph of J, for it includes all the edges and vertices of B not belonging to J. Moreover it is J-bounded, for any vertex of attachment of B' not belonging to J would be a vertex of attachment of B. Hence $B' \in \mathbf{J}$. By the definition of a bridge it follows that $B' = B$, that is E and V are null.

(2.2) *If x is any edge or vertex of G not belonging to J then x belongs to just one bridge of J in G.*

Proof. x belongs to at least one $H \in \mathbf{J}$, for example G. Choose such an H so that $\alpha_0(H) + \alpha_1(H)$ has the least possible value. Assume H is not a bridge of J in G. Then H has a proper subgraph $K \in \mathbf{J}$. By the definition of H, x is not an edge or vertex of K.

Now any end v of an edge of $E(H) - E(K)$ belongs either to $V(H) - V(K)$ or to $V(K)$. But if $v \in V(K)$ then v is a vertex of attachment of K and so $v \in V(J)$. Hence there is a subgraph F of H such that

$$E(F) = E(H) - E(K),$$
$$V(F) = (V(H) - V(K)) \cup (V(J) \cap V(K)).$$

F is J-bounded. For let w be any vertex of attachment of F not in $V(J)$. Since w is not a vertex of attachment of H it must be incident with an edge of K. But then $w \in V(K)$. This is contrary to the definition of $V(F)$. As x belongs to F we deduce that $F \in \mathbf{J}$. But

$$\alpha_0(F) + \alpha_1(F) < \alpha_0(H) + \alpha_1(H),$$

since K has at least one edge or vertex not belonging to J. This contradicts the definition of H.

We deduce that x belongs to some bridge H of J in G. Suppose it also belongs to a bridge H' of J in G. Then $H \cap H'$ is J-bounded and is not a subgraph of J. Hence $H = H \cap H' = H'$, by the definition of a bridge.

(2.3) *Let $P = (v_0, e_1, v_1, \ldots, v_k)$ be a non-degenerate path in G avoiding J. Then P is a path in some bridge B of J in G.*

Proof. Let B be the bridge of J which includes e_1 (2.2). If possible let e_i be the first edge-term of P not belonging to $E(B)$. Then $i > 1$ and v_{i-1} is a vertex of attachment of B. Hence $v_{i-1} \in V(J)$, contrary to the hypothesis that P avoids J. We deduce that P is a path in B.

(2.4) *Let v and w be vertices of the same bridge B of J in G. Then there is a simple path P from v to w in B avoiding J.*

Proof. Let X be the set of all $x \in V(B)$ such that there is a simple path

P_{vx} from v to x in B avoiding J. Let C be the subgraph of B such that $V(C) = X$ and $E(C)$ is the set of all edges of B having both ends in X.

Now $v \in X$. If $v \in V(J)$ it is incident with an edge e of B not in $E(J)$, by (2.1). Either e is a loop or there is a simple path in B of the form (v, e, v'). In each case $e \in E(C)$. Hence C is not a subgraph of J.

Suppose u is a vertex of attachment of C not in $V(J)$. Since it is not a vertex of attachment of B it is incident with some edge e' of $E(B)-E(C)$. The other end, t say, of e' is not in $V(C)$. It follows that t is not a term of the simple path P_{vu}. Hence we can obtain a simple path Q from v to t in B, avoiding J, by adjoining two extra terms, e' and t, to P_{vu}. But then $t \in V(C)$, a contradiction.

We deduce that C is J-bounded. Accordingly $C \in \mathbf{J}$. Hence $C = B$, by the definition of a bridge. Hence $w \in X$, and the theorem is proved.

The bridges of the null subgraph of G are often called the *components* of G. The graph G is said to be *connected* if it has just one component.

If P is any path in G then $G(P)$ is a connected graph, by (2.3) with $G = G(P)$. In particular arc-graphs and circuits are connected. A connected graph with no edge has just one vertex. We call it a *vertex-graph*. A connected graph with just one edge either consists of a loop with its single incident vertex, or of a link with its two ends. We call it a *loop-graph* or *link-graph* respectively.

We call G *separable* if it has two proper subgraphs H and K such that $H \cup K = G$ and $H \cap K$ is either null or a vertex-graph. Clearly every non-separable graph which is non-null is connected. Vertex-graphs, link-graphs, and loop-graphs are non-separable. Moreover any non-separable graph having a loop is necessarily a loop-graph.

If G is a circuit it is non-separable. For suppose H and K are as above. Since G is connected H and K have each at least one vertex of attachment. It follows that H and K have a common vertex v, and v is incident with at least one edge in both H and K. Since each vertex of G has degree 2 we have $d(K, v) = 1$, and $d(K, u) = 2$ for each $u \in V(K)-\{v\}$. This is contrary to (1.1).

(2.5) *Suppose G is non-separable, and not a link-graph. Then each edge of G belongs to some circuit of G.*

Proof. Let e be any edge of G. If e is a loop the theorem is evidently satisfied. If e is a link there exists a bridge B of $G \cdot \{e\}$. Since G is non-separable B has at least two vertices of attachment, that is B includes the two ends, x and y say, of e. There is a simple path P from x to y in B, by (2.4). Adjoining to P the extra terms e and x we obtain a circular path in G involving e.

(2.6) *Let H be a non-separable subgraph of G, and let A be an arc-graph of G spanning H. Then $H \cup A$ is non-separable.*

Proof. Suppose $H \cup A$ is separable. Then it has proper subgraphs K and L such that $K \cup L = H \cup A$ and $K \cap L$ is either null or a vertex-graph. Since H is non-separable we may suppose $H \subseteq K$. Now the ends of A are the ends of an arc-graph A' in the connected graph H, by (2.4), and $A \cup A'$ is a circuit. Since $A \cup A'$ is non-separable and $A' \subseteq H \subseteq K$ we must have $A \cup A' \subseteq K$. But then $H \cup A \subseteq K$, contrary to the definition of K as a proper subgraph of $H \cup A$.

3. Planar meshes

(3.1) *Let S be an elementary cycle of G. Then no proper subset of S is an elementary cycle of G.*

Proof. Suppose T is an elementary cycle of G such that $T \subseteq S$. Each vertex of $G \cdot T$ has degree 2 in both $G \cdot S$ and $G \cdot T$. Hence $G \cdot T$, regarded as a subgraph of $G \cdot S$, has no vertex of attachment. This is impossible since $G \cdot S$ is connected.

(3.2) *Let K be any non-null cycle of G. Then we can write*

$$K = \sum_{i=1}^{k} C_i,$$

where the C_i are disjoint elementary cycles of G satisfying $C_i \subseteq K$.

Proof. If possible choose K so that the theorem fails and K has the least number of edges consistent with this.

Suppose $G \cdot K$ is separable. Then $G \cdot K = (G \cdot K') \cup (G \cdot K'')$, where K' and K'' are proper subsets of K such that $(G \cdot K') \cap (G \cdot K'')$ is either null or a vertex-graph. But then K' and K'' are cycles of G, by (1.1). It follows by the definition of K that K' and K'' are sums of disjoint elementary cycles of G contained in $K' \cup K''$. But then the theorem is true for K, contrary to assumption.

We deduce that $G \cdot K$ is non-separable. Choose $e \in K$. By (2.5) there is an elementary cycle C_1 of G such that $e \in C_1 \subseteq K$. It follows that the theorem is true for K, for if the cycle $K + C_1$ is non-null it has fewer edges than K and is therefore a sum of disjoint elementary cycles C_i of G ($i = 2, \ldots, k$) satisfying $C_i \subseteq K + C_1 \subset K$. This contradiction establishes the theorem.

We go on to study the planar meshes of a graph G. To begin with we note that any vertex-graph or link-graph has a null planar mesh, and that any circuit C has a planar mesh $\{E(C), E(C)\}$.

For each $v \in V(G)$ we denote the set of incident edges of G by $S(G, v)$.

(3.3) *Suppose G is non-separable. Let* M *be a planar mesh and v a vertex* *of G. Let X and Y be complementary non-null subsets of $S(G, v)$. Then there exists $S_i \in \mathbf{M}$ such that $S_i \cap X$ and $S_i \cap Y$ have each just one edge.*

Proof. Since X and Y are non-null we have $\alpha_1(G) \geqslant 2$. Hence G has no loop. Choose $x \in X$ and $y \in Y$. Let the ends of x and y other than v be a and b respectively.

The vertex-graph defined by v has only one bridge in G, for otherwise G would be separable. By (2.4) there is a simple path P from a to b in G avoiding this vertex-graph. Adjoining the extra terms y, v, x, a to P we obtain a circular path from a to a in G. Hence there exists an elementary cycle C of G such that $C \cap X$ has just one edge.

Now C is a mod 2 sum of members of $\mathbf{M}$. Hence there exists $S_i \in \mathbf{M}$ such that the number of members of $S_i \cap X$ is odd. But $d(G \cdot S_i, v) = 2$. Hence $S_i \cap X$ and $S_i \cap Y$ have each just one edge.

Let $\mathbf{M}$ be a planar mesh of G, and let $\mathbf{U}$ be any subset of $\mathbf{M}$. We define a *decomposition* of $\mathbf{U}$ as a pair $\{\mathbf{X}, \mathbf{Y}\}$ of complementary non-null subsets of $\mathbf{U}$ such that no edge of G belongs both to a member of $\mathbf{X}$ and a member of $\mathbf{Y}$. If $\mathbf{U}$ has no decomposition we call it *connected.*

If $\mathbf{M} = \{S_1, S_2,..., S_k\}$ is connected and $k \geqslant 3$ then all the sets S_i are distinct. If for example $S_1 = S_2$, then the pair $\{\{S_1, S_2\}, \{S_3,..., S_k\}\}$ is a decomposition of $\mathbf{M}$.

If $\mathbf{W}$ is any class of subsets of $E(G)$, not necessarily all distinct, we denote the union and mod 2 sum of the members of $\mathbf{W}$ by $\bigcup \mathbf{W}$ and $\sum \mathbf{W}$ respectively.

(3.4) *If G is non-separable then every planar mesh of G is connected.*

Proof. Suppose a planar mesh $\mathbf{M}$ of G has a decomposition $\{\mathbf{X}, \mathbf{Y}\}$. Since G is connected there is a vertex v of attachment of $G \cdot \bigcup \mathbf{X}$. The sets $S(G, v) \cap \bigcup \mathbf{X}$ and $S(G, v) \cap \bigcup \mathbf{Y}$ are non-null and no member of $\mathbf{M}$ meets both of them. This is contrary to (3.3).

(3.5) *Let C be an elementary cycle of a non-separable graph G having a planar mesh $\mathbf{M}$. Then there are just two subsets $\mathbf{U}$ and $\mathbf{V}$ of $\mathbf{M}$ the* mod 2 *sum of whose members is C. Moreover $\mathbf{U}$ and $\mathbf{V}$ are connected complementary non-null subsets of $\mathbf{M}$.*

Proof. By Condition (ii) of the definition of a planar mesh there exists $\mathbf{U} \subseteq \mathbf{M}$ such that $\sum \mathbf{U} = C$. Let $\mathbf{V}$ be the complement of $\mathbf{U}$ in $\mathbf{M}$. Then $\sum \mathbf{V} = C$, by Condition (i). Clearly $\mathbf{U}$ and $\mathbf{V}$ are non-null.

Let $\mathbf{W}$ be any subset of $\mathbf{M}$ such that $\sum \mathbf{W} = C$. Then $\sum (\mathbf{W}+\mathbf{U})$ is null. Hence $\mathbf{W}+\mathbf{U}$ is either $\mathbf{M}$ or its null subset, for otherwise

$$\{\mathbf{W}+\mathbf{U}, \mathbf{M}-(\mathbf{W}+\mathbf{U})\}$$

would be a decomposition of $\mathbf{M}$, contrary to (3.4). Hence $\mathbf{W}$ can be only $\mathbf{U}$ or $\mathbf{V}$.

Suppose $\mathbf{U}$ has a decomposition $\{\mathbf{X}, \mathbf{Y}\}$. Then $\sum \mathbf{X} \subseteq C$. But $\sum \mathbf{X}$ is a cycle of G. Hence $\sum \mathbf{X} = \emptyset$ or C, by (3.1) and (3.2). Hence either $\sum \mathbf{X}$ or $\sum \mathbf{Y}$ is C, which contradicts the preceding result. Accordingly $\mathbf{U}$, and similarly $\mathbf{V}$, is connected.

We call the pair $\{\mathbf{U}, \mathbf{V}\}$ the *partition* of $\mathbf{M}$ determined by C. We also refer to $G \cdot \bigcup \mathbf{U}$ and $G \cdot \bigcup \mathbf{V}$ as the *residual graphs* of the circuit $G \cdot C$, with respect to $\mathbf{M}$.

(3.6) *Let G be a non-separable graph having a planar mesh* $\mathbf{M}$. *Let H be a circuit of G having residual graphs R and R' with respect to M. Then*

$$R \cup R' = G \quad \textit{and} \quad R \cap R' = H.$$

Proof. By (2.5) we have $E(R \cup R') = E(G)$. Hence $R \cup R' = G$ since G can have no isolated vertex.

By the definition of R and R' we have $E(H) = E(R \cap R')$, and therefore $V(H) \subseteq V(R \cap R')$. But suppose $R \cap R'$ has a vertex v not belonging to $V(H)$. Then no member of $\mathbf{M}$ meets both

$$S(G, v) \cap E(R) \quad \text{and} \quad S(G, v) \cap E(R'),$$

which contradicts (3.3). We deduce that $V(H) = V(R \cap R')$. Hence

$$H = R \cap R'.$$

Let v and w be distinct vertices of a circuit K. It follows from the definitions of arc-graphs and circuits that K is the union of two arc-graphs A_1 and A_2 which have the same ends v and w, but which have no edge or internal vertex in common. They are uniquely determined as the bridges in K of the edgeless subgraph J defined by $V(J) = \{v, w\}$. We call them the *residual arc-graphs* of v and w in K.

(3.7) *Let G be a non-separable graph having a planar mesh* $\mathbf{M}$. *Let C be an elementary cycle of G determining the partition* $\{\mathbf{U}, \mathbf{V}\}$ *of* $\mathbf{M}$. *Let A be an arc-graph of $G \cdot \bigcup \mathbf{U}$ spanning $G \cdot C$. Let A_1 and A_2 be the residual arc-graphs of the ends of A in $G \cdot C$. Then there exist complementary non-null subsets* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *of* $\mathbf{U}$ *such that* $\sum \mathbf{X}_1 = E(A \cup A_1)$ *and* $\sum \mathbf{X}_2 = E(A \cup A_2)$.

Proof. $E(A \cup A_1)$ is an elementary cycle of G. Let it determine the partition $\{\mathbf{U}_1, \mathbf{V}_1\}$ of $\mathbf{M}$. No edge of G belongs both to $\bigcup (\mathbf{U}_1 \cap \mathbf{V})$ and to $\bigcup (\mathbf{V}_1 \cap \mathbf{V})$. For suppose e is such an edge. We have

$$e \in \bigcup \mathbf{U}_1 \cap \bigcup \mathbf{V}_1 \cap \bigcup \mathbf{V} = (E(A) \cup E(A_1)) \cap \bigcup \mathbf{V} \subseteq C.$$

But each edge of C belongs to just one member of $\mathbf{V}$ and therefore cannot belong to both $\bigcup (\mathbf{U}_1 \cap \mathbf{V})$ and $\bigcup (\mathbf{V}_1 \cap \mathbf{V})$. Since $\mathbf{V}$ is connected we deduce that one of the sets $\mathbf{U}_1 \cap \mathbf{V}$ and $\mathbf{V}_1 \cap \mathbf{V}$ is null. Without loss of generality we may suppose $\mathbf{U}_1 \cap \mathbf{V}$ is null, that is $\mathbf{U}_1 \subseteq \mathbf{U}$.

We have seen that there is a non-null subset $\mathbf{X}_1 = \mathbf{U}_1$ of $\mathbf{U}$ such that $\sum \mathbf{X}_1 = E(A \cup A_1)$. Similarly there is a non-null subset $\mathbf{X}_2$ of $\mathbf{U}$ such that $\sum \mathbf{X}_2 = E(A \cup A_2)$. But then $\sum (\mathbf{X}_1 + \mathbf{X}_2) = C$. Hence $\mathbf{X}_1 + \mathbf{X}_2 = \mathbf{U}$, by (3.5). Accordingly the subsets $\mathbf{X}_1$ and $\mathbf{X}_2$ of $\mathbf{U}$ are complementary.

4. 3-linkage

(4.1) *Let* $\mathbf{M} = \{S_1, \ldots, S_k\}$ *be a planar mesh of a non-separable graph* G *which is 3-linked with respect to* S_1. *Then if* $2 \leqslant i < j \leqslant k$ *the intersection* $(G \cdot S_i) \cap (G \cdot S_j)$ *is null, a vertex-graph, or an arc-graph* $G \cdot L$. *In the last case* $G \cdot (S_i + S_j)$ *is a circuit spanned by* $G \cdot L$, *and* $G \cdot (S_i - L)$ *and* $G \cdot (S_j - L)$ *are the residual arc-graphs of the ends of* $G \cdot L$ *in* $G \cdot (S_i + S_j)$.

Proof. $\mathbf{M}$ is connected, by (3.4). Hence, since $k \geqslant 3$, the members of $\mathbf{M}$ are all distinct. Write $H = (G \cdot S_i) \cap (G \cdot S_j)$.

$G \cdot S_i$ is not a subgraph of $G \cdot S_j$, by (3.1). Hence there is a bridge B of $G \cdot S_j$ in $G \cdot (S_i \cup S_j)$. If B has fewer than 2 vertices of attachment then $B = G \cdot S_i$, since otherwise $G \cdot S_i$ would be separable. In this case H is either null or a vertex-graph.

In the remaining case $G \cdot S_j$ is spanned by an arc-graph A of $G \cdot S_i$, by (2.4). Let A' be one of the residual arc-graphs in $G \cdot S_j$ of the ends of A. Then $E(A \cup A')$ is an elementary cycle of G meeting both S_i and S_j. Moreover we can choose A' so that $E(A \cup A')$ is not S_i.

By the foregoing argument G has an elementary cycle C, distinct from S_i and S_j, such that $C \subseteq S_i \cup S_j$ and C meets both S_i and S_j. Any such C determines a partition $\{\mathbf{U}, \mathbf{V}\}$ of $\mathbf{M}$, where we may suppose that either S_i or S_j belongs to $\mathbf{U}$. We choose C and $\mathbf{U}$ so that $\mathbf{U}$ has as few members as possible. If possible we arrange also that both S_i and S_j belong to $\mathbf{U}$. Having chosen C and $\mathbf{U}$ we adjust the notation, by interchanging the suffixes of S_i and S_j if necessary, so that $S_i \in \mathbf{U}$.

By (3.1) there is a bridge B' of $G \cdot C$ in $G \cdot (C \cup S_i)$. This has at least two vertices of attachment, for the non-separable graph $G \cdot S_i$ has an edge in common with $G \cdot C$. By (2.4) there is an arc-graph $G \cdot L$ of $G \cdot S_i$ spanning $G \cdot C$. Let the ends of $G \cdot L$ be x and y, and let their residual arc-graphs in $G \cdot C$ be $G \cdot L'$ and $G \cdot L''$.

By (3.5) and (3.7) the elementary cycles $L \cup L'$ and $L \cup L''$ of G determine partitions $(\mathbf{U}', \mathbf{V}')$ and $(\mathbf{U}'', \mathbf{V}'')$ of $\mathbf{M}$ respectively, where $\mathbf{U}'$ and $\mathbf{U}''$ are complementary non-null subsets of $\mathbf{U}$. Without loss of generality we may suppose $S_i \in \mathbf{U}'$.

We have $\mathbf{U}' = \{S_i\}$ and $S_i = L \cup L'$, since otherwise the choice of C and

$\mathbf{U}$ is contradicted. Similarly if $S_j \in \mathbf{U}''$ we have $S_j = L \cup L''$. But then the theorem holds, with $G\cdot(S_i+S_j) = G\cdot(L'+L'') = G\cdot C$. We may therefore assume $S_j \notin \mathbf{U}''$.

Let R be the residual graph $G\cdot \bigcup \mathbf{U}''$ of the circuit $G\cdot(L \cup L'')$. Let v be any vertex of R other than x and y. We recall that $L \subseteq S_i$ and $L'' \subseteq S_j$. Now each edge of $S(R,v)$ belongs to two members of the set $\mathbf{U}'' \cup \{S_i, S_j\}$. Moreover each member of this set contains either none or just two of the edges of $S(R, v)$. It follows from (3.3) and the definition of a planar mesh that $S(R,v) = S(G,v)$.

We deduce that x and y are the only vertices of attachment of R in G. Hence

$$\alpha_0((G\cdot \bigcup \mathbf{U}'') \cap (G\cdot(E(G)- \bigcup \mathbf{U}''))) = 2.$$

Since G is 3-linked with respect to S_1 this implies that

$$S_1 \in \mathbf{U}''$$

and $E(G)- \bigcup \mathbf{U}''$ contains no elementary cycle of G. But then $\mathbf{V}'' = \{S_i, S_j\}$. This is contrary to the choice of C and $\mathbf{U}$, for $\mathbf{U}$ has at least two members and $S_j \notin \mathbf{U}$. The theorem follows.

The *nodes* of a graph G are those vertices v of G for which $d(G, v) \neq 2$. They constitute a subset $N(G)$ of $V(G)$. The *branches* of G are the bridges in G of the edgeless subgraph J satisfying $V(J) = N(G)$. A branch B including two distinct vertices x and y is an arc-graph spanning J and with ends x and y. For B has such an arc-graph as a subgraph, by (2.4), and this arc-graph is J-bounded. A branch not including two distinct nodes of G is a circuit. For it has no vertex of odd degree, by (1.1), and so has a circuit, by (3.2). This circuit is J-bounded. If G is non-separable and is not a circuit, then all its branches are arc-graphs spanning J.

(4.2) *Let C be an elementary cycle of a graph G. Let H be a non-separable subgraph of G such that $C \subseteq E(H)$ and H is 3-linked with respect to C. Let K be an arc-graph or Y-graph of G spanning H. Then either K spans a branch B of H which is not a subgraph of $G\cdot C$, or $H \cup K$ is non-separable and 3-linked with respect to C.*

Proof. By one or two applications of (2.6) we find that $H \cup K$ is non-separable.

Assume $H \cup K$ is not 3-linked with respect to C. Then there exist complementary non-null subsets S and S' of $E(H \cup K)$ with the following properties.

(i) $\alpha_0((G\cdot S) \cap (G\cdot S')) = 2$,

(ii) $C \subseteq S$,

(iii) $G\cdot S'$ has at least one circuit.

We denote the common vertices of $G\cdot S$ and $G\cdot S'$ by x and y.

An arc-graph A can have no circuit as a subgraph. (The circuit would be a bridge distinct from A of the null subgraph of the connected graph A.) Moreover a Y-graph can have no circuit. For such a circuit, being non-separable, would be a subgraph of one of the arms of the Y-graph. Hence we can find $s \in E(H) \cap S$ and $s' \in E(H) \cap S'$.

There is a bridge D of $H \cap (G \cdot S)$ in H such that $s' \in E(D)$, by (2.2). Since H is non-separable, D has both x and y as vertices. Hence, by (2.4), there is an arc-graph L' of $H \cap (G \cdot S')$, with ends x and y, spanning $H \cap (G \cdot S)$. Similarly there is an arc-graph L of $H \cap (G \cdot S)$, with ends x and y, spanning $H \cap (G \cdot S')$.

Let D_1 be a bridge of the circuit $L \cup L'$ in either H or $H \cup K$. Then $E(D_1)$ does not meet both S and S'. For otherwise $D_1 \cap (G \cdot S)$ and $D_1 \cap (G \cdot S')$ are not $(L \cup L')$-bounded and they therefore have a common vertex other than x and y. This is contrary to (i). Hence either $D_1 \subseteq G \cdot S$ or $D_1 \subseteq G \cdot S'$.

Suppose D_1 is a bridge in H. It has two distinct vertices u and v of attachment, since H is non-separable. If $D_1 \subseteq G \cdot S'$ these are both vertices of L'. They are the ends of an arc-graph L_1 of D_1 spanning $L \cup L'$, and the ends of an arc-graph L_2 of the connected graph L'. But then $L_1 \cup L_2$ is a circuit of $H \cap (G \cdot S')$. This is impossible since H is 3-linked with respect to C. We deduce that $H = L' \cup (H \cap (G \cdot S))$. This implies that each internal vertex of L' is of degree 2 in H. Hence L' is a subgraph of a branch B of H. Clearly B is not a subgraph of $G \cdot C$.

By (iii) there is at least one bridge of $L \cup L'$ in $H \cup K$ which is a subgraph of $G \cdot S'$. By the above considerations this must be a subgraph of K. Hence it is identical with K, by (2.2) and (2.3). Accordingly K spans B.

(4.3) *Let B be a bridge of a subgraph J in a graph G. Let x, y, and z be distinct common vertices of B and J. Then there is a Y-graph Y of B, with ends x, y, and z, which spans J.*

Proof. By (2.4) there is a simple path P from x to y in B avoiding J. If P includes only one edge e, then the proper subgraph $G \cdot \{e\}$ of B is J-bounded, contrary to the definition of a bridge. Hence P includes a vertex u distinct from x and y. By (2.4) there is a simple path Q from z to u in B avoiding J. Let u' be the first vertex of P occurring in Q, and let Q_1 be the part of Q extending from z to u'. Let Q_2 be the part of P extending from x to u', and Q_3 the part of P extending from u' to y. Then $G(Q_1)$, $G(Q_2)$, and $G(Q_3)$ are the arms of a Y-graph Y of B, with centre u' and ends x, y, and z, which spans J.

(4.4) *Let G be a non-separable graph, 3-linked with respect to an elementary* *cycle C. Then either $G = G \cdot C$ or $G = H \cup A$, where $C \subseteq E(H)$, H is non-separable and 3-linked with respect to C, and A is an arc-graph spanning H.*

Proof. Assume $G \neq G \cdot C$. Then we can find a proper subgraph K of G such that $C \subseteq E(K)$, K is non-separable, and K is 3-linked with respect to C. For example we can take $K = G \cdot C$. We choose K so that $\alpha_1(K)$ has the greatest possible value.

We note that all the vertices of attachment of $G \cdot C$ in K are nodes of K. Hence $G \cdot C$ is a union of branches of K. Accordingly any branch of K which is not a subgraph of $G \cdot C$ has no edge or internal vertex in common with $G \cdot C$.

Since G is non-separable each bridge of K in G has at least two vertices of attachment.

We suppose first that to each bridge B of K in G there corresponds a branch $Z(B)$ of K, not a subgraph of $G \cdot C$, such that $V(Z(B))$ includes all the vertices of attachment of B. In this case we select a bridge B' of K in G and write U for the union of $Z(B')$ and all bridges B of K in G such that $Z(B) = Z(B')$. Then $G \cdot E(U)$ and $G \cdot (E(G) - E(U))$ have only two common vertices, the ends of $Z(B')$. But $C \subseteq E(G) - E(U)$ and $G \cdot E(U)$ has at least one circuit, by (2.4). This contradicts the hypothesis that G is 3-linked with respect to C.

We deduce that there is a bridge B of K in G with the following property: no branch Z of K which is not a subgraph of $G \cdot C$ includes all the vertices of attachment of B.

Suppose x and y are distinct vertices of attachment of B. By (2.4) there is an arc-graph L of B with ends x and y which spans K. Assume that no branch of K which is not a subgraph of $G \cdot C$ has both x and y as vertices. Then, by (4.2), $K \cup L$ is non-separable and 3-linked with respect to C. But then $G = K \cup L$, by the definition of K. Accordingly the theorem is true with $H = K$ and $A = L$.

We may now suppose that to each pair $\{x, y\}$ of distinct vertices of attachment of B there corresponds a branch $Z(x, y)$ of K, not a subgraph of $G \cdot C$, such that $\{x, y\} \subseteq V(Z(x, y))$. It follows from the definition of B that B has at least three vertices of attachment.

We cannot choose x, y and $Z(x, y)$ so that x is an internal vertex of the arc-graph $Z(x, y)$. For if this were so $Z(x, y')$ would have to be identical with $Z(x, y)$ for each vertex of attachment y' of B other than x. This would contradict the definition of B. We deduce that $Z(x, y)$ is an arc-graph with ends x and y.

Choose three vertices of attachment x, y, and z of B. By (4.3) there is a Y-graph Y of B, with ends x, y, and z, spanning K. Let its centre be u. Let its arms be A_x, A_y, and A_z, with ends x, y, and z respectively. By the foregoing results Y spans no branch of K which is not a subgraph of $G \cdot C$. Hence, by (4.2), $K \cup Y$ is non-separable and 3-linked with respect to C. Hence $G = K \cup Y$, and $B = Y$, by the definition of K.

Write $T = Y \cup Z(x,y) \cup Z(x,z) \cup Z(y,z)$. We note that T includes no edge of $G \cdot C$. Hence there exists a bridge B' of T in G. Since G is non-separable we may suppose that x and y are vertices of B'. By (2.4) there is an arc-graph M of B', with ends x and y, which spans T. We note that no two of the arc-graphs M, $A_x \cup A_y$, and $Z(x,z) \cup Z(y,z)$ have any edge or internal vertex in common. We write K' for the subgraph of G obtained by removing the edges and internal vertices of $Z(x,y)$.

Suppose K' is separable. Then there are proper subgraphs P and Q of K' such that $K' = P \cup Q$, and $P \cap Q$ is either null or a vertex-graph. Since G is not separable we may suppose that x is a vertex of P but not of Q, and that y is a vertex of Q but not of P. But then each of the arc-graphs M, $A_x \cup A_y$, and $Z(x, z) \cup Z(y, z)$ must have an internal vertex which is common to P and Q, contrary to the condition on $P \cap Q$. We deduce that K' is non-separable.

Suppose K' is not 3-linked with respect to C. Then we can write

$$K' = (G \cdot S) \cup (G \cdot S'),$$

where $C \subseteq S$, $G \cdot S'$ has at least one circuit, and $G \cdot S$ and $G \cdot S'$ have just two common vertices and no common edge. Since G is 3-linked with respect to C we may suppose x is a vertex of $G \cdot S$ but not of $G \cdot S'$, and y is a vertex of $G \cdot S'$ but not of $G \cdot S$. But then each of the arc-graphs M, $A_x \cup A_y$, and $Z(x,z) \cup Z(y,z)$ must have an internal vertex common to $G \cdot S$ and $G \cdot S'$, contrary to the definition of $G \cdot S$ and $G \cdot S'$. We deduce that K' is 3-linked with respect to C.

We observe that the theorem is true with $H = K'$ and $A = Z(x,y)$.

5. Convex representations

In this section we prove the two Theorems I and II stated in the Introduction.

Proof of Theorem I. Write $\mathbf{M} = \{C_1,\dots, C_k\}$.

Each edge of R belongs to the boundaries of just two polygons P_i. Hence $\mathbf{M}$ satisfies Condition (i) for a planar mesh. Now let C be any elementary cycle of G. The corresponding set of edges of R determines a polygon P in Π. Let $\mathbf{U}$ be the set of all $C_i \in \mathbf{M}$ such that Q_i lies inside P. Then $C = \sum \mathbf{U}$. Hence, by (3.2), $\mathbf{M}$ satisfies Condition (ii) for a planar mesh.

The planar mesh $\mathbf{M}$ is connected and G is non-separable. For otherwise G is separable, by (3.4). Then G is the union of proper subgraphs H and K

having at most one vertex in common. Since circuits are non-separable we may suppose $C_1 \subseteq H$. Choose $x \in V(H)$, taking $x \in V(K)$ if this is possible. Choose $y \in V(K)$ so that the segment $f(x)f(y)$ is as long as possible. Then one of the regions Q_i $(2 \leqslant i \leqslant k)$ occupies a reflex angle at $f(y)$, contrary to the definition of R.

Let S and T be complementary non-null subsets of $E(G)$ such that $C_1 \subseteq S$ and the graphs $G \cdot S$ and $G \cdot T$ have only two vertices x and y in common. Choose $z \in V(G \cdot T)$ so that the distance d of $f(z)$ from the segment $f(x)f(y)$ is as great as possible. Then if $d > 0$ one of the regions Q_i $(2 \leqslant i \leqslant k)$ occupies a reflex angle at $f(z)$, contrary to the definition of R. Hence each vertex of $G \cdot T$ corresponds to a point of the segment $f(x)f(y)$ under the mapping f.

We deduce from the above result that $G \cdot T$ can have no circuit. Hence G is 3-linked with respect to C_1. We deduce also that x and y cannot be the two ends of an edge of C_1. Applying this result to the case in which $G \cdot T$ is a branch of G spanning $G \cdot C_1$ we complete the proof of the theorem.

Proof of Theorem II. Let $q(C_1)$ be the number of edges of C_1 whose ends are both nodes of G. We discuss first the case $q(C_1) = 0$.

M is non-null, for $C_1 \in \mathbf{M}$. Hence $k \geqslant 2$. If $k = 2$ then $\mathbf{M} = \{C_1, C_2\}$ and G is the circuit $G \cdot C_1$, by (2.6). In this case the theorem holds, with $f = h$.

Assume as an inductive hypothesis that the theorem is true whenever $q(C_1) = 0$ and k is less than some integer $n > 2$. Consider the case in which $q(C_1) = 0$ and $k = n$.

By (4.4) we may now write $G = H \cup A$, where $C_1 \subseteq E(H)$, H is non-separable and 3-linked with respect to C_1, and A is an arc-graph spanning H. Since $q(C_1) = 0$ no branch of H spanning $H \cdot C_1$ can have the same ends as any edge of C_1.

Choose $e \in E(A)$. By (2.5) we may suppose $e \in C_{n-1} \cap C_n$. Since each internal vertex of A has degree 2 in G we have $A \subseteq (G \cdot C_{n-1}) \cap (G \cdot C_n)$. But by (4.1) and (3.6) this intersection is a branch of G spanning the circuit $G \cdot (C_{n-1} + C_n)$. Since H is non-separable it follows that

$$A = (G \cdot C_{n-1}) \cap (G \cdot C_n).$$

Write $\mathbf{M}' = \{C_1, \ldots, C_{n-2}, C_{n-1} + C_n\}$. Then the members of $\mathbf{M}'$ are elementary cycles of H, and $\mathbf{M}'$ satisfies Condition (i) for a planar mesh. But any cycle of H is a mod 2 sum of members of $\mathbf{M}'$. For it is a mod 2 sum of members of $\mathbf{M}$ and does not have e as a member. Hence $\mathbf{M}'$ is a planar mesh of H.

It now follows, by the inductive hypothesis, that there is a 1–1 mapping

f' of $V(H)$ onto a set of points of Π such that the following conditions are satisfied:

(i) $f'(x) = h(x)$ if $x \in V(G \cdot C_1)$,

(ii) f' determines a non-singular convex representation R' of H, with respect to which H has the frame C_1 and the outer cycles

$$C_1, \ldots, C_{n-2}, C_{n-1}+C_n.$$

We denote the convex polygons corresponding to these outer cycles by $P_1, \ldots, P_{n-2}$, P respectively.

Let u and v be the ends of A. Let A_{n-1} and A_n be the residual arc-graphs of u and v other than A in $G \cdot C_{n-1}$ and $G \cdot C_n$ respectively. By (4.1) these are also the residual arc-graphs of u and v in the circuit $G \cdot (C_{n-1}+C_n)$.

It follows that $f'(u)$ and $f'(v)$ belong to the boundary of P. Let s be the segment joining them. If s lies in an edge of P then, since R' is non-singular, A spans a branch B of H whose vertices correspond to points of s. If B is not a subgraph of $G \cdot C_1$ this contradicts the hypothesis that G is 3-linked with respect to C_1. If B is a subgraph of $G \cdot C_1$, then s must be an edge of P_1. But then both ends of the corresponding edge of $G \cdot C_1$ are nodes of G, contrary to the assumption that $q(C_1) = 0$. We deduce that s crosses the interior of P, thus separating P into two convex polygons P_{n-1} and P_n. We may suppose that the boundaries of P_{n-1} and P_n include the edges of the arc-graphs of R' corresponding to A_{n-1} and A_n respectively.

Let the vertices of A, taken in order from u to v, be $x_1 = u, x_2, \ldots, x_r = v$. Let $X_1 = f'(u), X_2, \ldots, X_r = f'(v)$ be distinct points, occurring in that order, on s. We define a mapping f of $V(G)$ onto a set of points of Π as follows. If $x \in V(H)$, then $f(x) = f'(x)$; if $x = x_i \in V(A)$, then $f(x) = X_i$.

We observe that f has the following properties.

(iii) $f(x) = h(x)$ if $x \in V(G \cdot C_1)$.

(iv) f determines a convex representation R of G, with respect to which G has the frame C_1 and the outer cycles $C_1, C_2, \ldots, C_n$.

(v) G has no singular node distinct from u and v with respect to R.

The convex polygons corresponding to $C_1, C_2, \ldots, C_n$ are $P_1, P_2, \ldots, P_n$ respectively.

Suppose u is a singular node. Then $f(u)$ is an interior point of some side of a polygon P_j, where $1 < j < n-1$. Let the ends of this side be $f(w)$ and $f(t)$. Choose a point Z on s between $f(u) = X_1$ and X_2. We modify the mapping f as follows. We replace $f(u)$ by Z, and we replace any $f(p)$ on the segment $f(u)f(w)$ or $f(u)f(t)$ by the point of intersection of the line through $f(p)$ parallel to $Zf(u)$ with the segment $Zf(w)$ or $Zf(t)$ respectively. Provided Z is sufficiently near to the original point $f(u)$ the mapping f will retain properties (iii), (iv), and (v). P_j will be augmented by the triangles $Zf(u)f(w)$ and $Zf(u)f(t)$, and P_{n-1} and P_n will be correspondingly reduced. We note also that u is non-singular with respect to the new representation. If necessary we perform a similar operation to make v non-singular.

We thus find that the theorem is true when $q(C_1) = 0$ and $k = n$. It follows by induction that the theorem is true whenever $q(C_1) = 0$.

Assume as a new inductive hypothesis that the theorem is true whenever $q(C_1)$ is less than some positive integer q. Consider the case $q(C_1) = q$. We can find $e \in C_1$ so that both ends of e, u and v say, are nodes of G. We construct a graph G' from G in the following way. We replace e by a new vertex w and two new edges e' and e''. The ends of e' are w and u, and the ends of e'' are w and v. Clearly the cycles of G' can be derived from those of G by replacing e, whenever it occurs, by the two edges e' and e''. If K is any cycle of G we denote the cycle of G' obtained from it by this operation by K'. We deduce from this correspondence that the class

$$\mathbf{M}' = \{C'_1, \ldots, C'_k\}$$

is a planar mesh of G'.

Construct a convex polygon P'_1 in Π having all the vertices of P_1 and just one other, W say, occurring between $h(u)$ and $h(v)$. Extend the mapping h by writing $h(w) = W$.

Now the value of $q(C'_1)$ for G' is less than q. So by the inductive hypothesis we can find a 1–1 mapping f of $V(G')$ onto a set of points of Π such that the following conditions are satisfied.

(vi) $f(x) = h(x)$ if $x \in V(G' \cdot C'_1)$.

(vii) f determines a non-singular convex representation R of G', with respect to which G' has the frame C'_1 and the outer cycles $C'_1, \ldots, C'_k$.

Let s denote the segment $f(u)f(v)$. Assume s is an edge of one of the convex polygons of R. Then there is an arc-graph A of G' whose vertices all correspond to points of s, and whose ends are u and v. Since R is non-singular, A must be a branch of G'. But then A is a branch of G spanning $G \cdot C_1$ and having the same ends as e, which is contrary to hypothesis. We deduce that s is a diagonal of that convex polygon P of R which has $h(w)$ as a vertex and is distinct from P'_1. But in this case f, when restricted to the vertices of G, determines a non-singular convex representation of G with the required properties. In this representation s corresponds to the edge e. The polygons are the same as those of R, except that P'_1 is augmented by the triangle $h(u)h(v)h(w)$ and P is correspondingly reduced.

We deduce that the theorem is true when $q(C_1) = q$. Hence it is true in general by induction.

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