Answers to Homework 2, Fall 2003

Question 1A

\[ F_{TA} \cdot F_{AC} \cdot F_{ST} = F_{TI} \cdot F_{IC} \]
\[ F_{AC} \cdot F_{ST} = F_{TA}^{-1} \cdot F_{TI} \cdot F_{IC} \]
\[ F_{AC} = F_{TA}^{-1} \cdot F_{TI} \cdot F_{IC} \cdot F_{ST}^{-1} \]

Question 1B

Define \( F_{TA}^{-1} \cdot F_{TI} = F_{AI} \). Then, the answer to Question 1A could be written

\[ F_{AC} = F_{AI} \cdot F_{IC} \cdot F_{ST}^{-1} \]

Now note that the position of the probe tip with respect to \( F_A \) is given by \( \mathbf{p}_{tip} = F_{AP} \cdot \mathbf{p}_{PT} \). Let \( F_{AP,i} \) be the coordinate system of the pointer with respect to the tracker base when the pointer probe is touching the \( i \)th pin. The corresponding tip position will be \( \mathbf{p}_{tip,i} = F_{AP,i} \cdot \mathbf{p}_{PT} \). Then we have the system of equations

\[ F_{AP,i} \cdot \mathbf{p}_{PT} = F_{AI} \cdot \mathbf{p}_{I,i} + \mathbf{p}_{AI} \]

for \( i = 1, 2, 3 \)

This may be expanded to

\[ \mathbf{p}_{tip,i} = R_{AI} \cdot \mathbf{p}_{I,i} + \mathbf{p}_{AI} \]

for \( i = 1, 2, 3 \)
Question 1C

Here we assume that we know $F_{AI}$ exactly. The surgeon has placed the osteotome so that

$$F_{AS,0} = F_{AC} = F_{AI} \cdot F_{IC} \cdot F_{ST}^{-1}$$

The nominal position of the tip with respect to $F_{AI}$ is given by

$$F_{AI}^{-1} \cdot F_{AS,0} \cdot F_{ST}$$

The actual position will be given by

$$F_{IC}^* = F_{AI}^{-1} \cdot F_{AS,0}^* \cdot F_{ST}$$

$$R_{IC}^* = R_{AI}^{-1} \cdot R_{AS,0} \cdot \Delta R_{AS,0} \cdot R_{ST}$$

$$\bar{p}_{IC}^* = F_{AI}^{-1} \cdot F_{AS,0} \cdot (\Delta R_{AS,0} \cdot \bar{p}_{ST} + \Delta \bar{p}_{AS,0})$$

$$= F_{AI}^{-1} \cdot F_{AS,0} \cdot \Delta \bar{p}_{AS,0} + F_{AI}^{-1} \cdot F_{AS,0} \cdot \Delta \bar{p}_{AS,0}$$

Question 1D

The rotational components will be given by
\[
\begin{align*}
R_{AI}^{-1} \cdot R_{AS,0} \cdot R_{ST} \cdot \Delta R_{IC} &= R_{AI}^{-1} \cdot R_{AS,0} \cdot \Delta R_{AS,0} \cdot R_{ST} \\
\Delta R_{IC} &= \left( R_{AI}^{-1} R_{AS,0} R_{ST} \right)^{-1} \cdot R_{AI}^{-1} R_{AS,0} \Delta R_{AS,0} R_{ST} \\
&= R_{ST}^{-1} \cdot \Delta R_{AS,0} \cdot R_{ST} \\
I + sk(\tilde{\alpha}_{IC}) &\approx R_{ST}^{-1} \cdot (I + sk(\tilde{\alpha}_0)) \cdot R_{ST} \\
&= I + R_{ST}^{-1} \cdot R_{ST} \cdot sk(R_{ST}^{-1} \tilde{\alpha}_0) \\
\tilde{\alpha}_{IC} &= R_{ST}^{-1} \tilde{\alpha}_0
\end{align*}
\]

The translational components will be

\[
\begin{align*}
\vec{p}_{IC}^* &= F_{AI}^{-1} \cdot F_{AS,0} \cdot \left( \Delta R_{AS,0} \cdot \vec{p}_{ST} + \Delta \vec{p}_{AS,0} \right) \\
&= R_{AI}^{-1} \cdot \left( R_{AS,0} \cdot \left( \Delta R_{AS,0} \cdot \vec{p}_{ST} + \Delta \vec{p}_{AS,0} \right) + \vec{p}_{AS,0} \right) - R_{AI}^{-1} \vec{p}_{AI} \\
&\approx R_{AI}^{-1} \cdot \left( \vec{p}_{ST} + sk(\tilde{\alpha}_0) \cdot \vec{p}_{ST} + \vec{\varepsilon}_0 \right) + \vec{p}_{AS,0} - R_{AI}^{-1} \vec{p}_{AI} \\
&= R_{AI}^{-1} \cdot R_{AS,0} \cdot \vec{p}_{ST} + R_{AI}^{-1} R_{AS,0} \left( \tilde{\alpha}_0 \times \vec{p}_{ST} + \vec{\varepsilon}_0 \right) \\
&= \vec{p}_{IC} + R_{AI}^{-1} R_{AS,0} \left( \tilde{\alpha}_0 \times \vec{p}_{ST} + \vec{\varepsilon}_0 \right)
\end{align*}
\]
This gives an uncertainty of

\[ \Delta \bar{p}_{IC} = R_{AI}^{-1} R_{AS,0} \left( \bar{\alpha}_0 \times \bar{p}_{ST} + \bar{\varepsilon}_0 \right) \]

\[ = R_{AI}^{-1} R_{AS,0} \left( \bar{\varepsilon}_0 - \bar{p}_{ST} \times \bar{\alpha}_0 \right) \]

\[ = R_{AI}^{-1} R_{AS,0} \bar{\varepsilon}_0 + R_{AI}^{-1} R_{AS,0} \bar{\alpha}_0 \cdot \bar{\varepsilon}_0 \]

**Question 1E**

The easiest way to approach this is to repeat the analysis above assuming that there is some registration error \( F_{AI}^* = F_{AI} \Delta F_{AI} \). Now the new actual pose \( F_{IC}^{**} \) of the osteotome with respect to \( F_{AI} \) will be given by

\[
\left( F_{AI}^* \right)^{-1} \cdot F_{AS,0}^* \cdot F_{ST} = \left( F_{AI} \Delta F_{AI} \right)^{-1} F_{AS,0}^* F_{ST}
\]

\[ = \Delta F_{AI}^{-1} F_{AI}^{-1} F_{AS,0}^* F_{ST} \]

\[ = \Delta F_{AI}^{-1} \cdot F_{IC}^* \]

where \( F_{IC}^* \) is the solution in Question 1E. The new actual position \( \bar{p}_{IC}^{**} \) will be given by
Thus we can compute bounds on the total positional uncertainty $\bar{\varepsilon}_{\text{tot}} = \bar{\varepsilon}_{\text{IC}}^* - \bar{\varepsilon}_{\text{IC}}$ from

$$\bar{\varepsilon}_{\text{tot}} = \bar{p}_{\text{IC}}^* = \Delta F_{\text{AI}}^{-1} \cdot \bar{p}_{\text{IC}}^* = \Delta R_{\text{AI}}^{-1} \cdot \left( \bar{p}_{\text{IC}}^* - \Delta \bar{p}_{\text{AI}} \right)$$

$$\approx \left( \mathbf{I} - sk(\bar{\alpha}_{\text{AI}}) \right) \cdot \left( \left( \bar{p}_{\text{IC}} + R_{\text{AI}}^{-1} R_{\text{AS,0}} \left( \bar{\alpha}_0 \times \bar{p}_{\text{ST}} + \bar{\varepsilon}_0 \right) \right) - \bar{\varepsilon}_{\text{AI}} \right)$$

$$= \bar{p}_{\text{IC}} + R_{\text{AI}}^{-1} R_{\text{AS,0}} \left( \bar{\alpha}_0 \times \bar{p}_{\text{ST}} + \bar{\varepsilon}_0 \right) - \bar{\varepsilon}_{\text{AI}}$$

$$+ \bar{\alpha}_{\text{AI}} \times \left( \bar{p}_{\text{IC}} + R_{\text{AI}}^{-1} R_{\text{AS,0}} \left( \bar{\alpha}_0 \times \bar{p}_{\text{ST}} + \bar{\varepsilon}_0 \right) - \bar{\varepsilon}_{\text{AI}} \right)$$

$$\approx \bar{p}_{\text{IC}} + R_{\text{AI}}^{-1} R_{\text{AS,0}} \left( \bar{\alpha}_0 \times \bar{p}_{\text{ST}} + \bar{\varepsilon}_0 \right) + \bar{\alpha}_{\text{AI}} \times \bar{p}_{\text{IC}} - \bar{\varepsilon}_{\text{AI}}$$

This can be simplified a bit to

$$F_{\text{AP},i} \cdot \left( \Delta R_{\text{AP},i} \cdot \bar{p}_{\text{PT}} + \Delta \bar{p}_{\text{AP},i} \right) = F_{\text{AI}} \cdot \left( \Delta R_{\text{AI}} \cdot \bar{p}_{\text{I},i} + \Delta \bar{p}_{\text{AI}} \right)$$

These linear equalities can be combined with the inequalities given in the question to produce the desired system of constraints.
Question 2

You can estimate the error as

\[(1000 \text{ mm}) \times 0.01 + 0.2 \text{ mm} \approx 10 \text{ mm}\]

Question 3

Essentially, you need to make \( F_B \) the base coordinate system for all your calculations. To do this make the substitutions

\[
\begin{align*}
F_{AP} &\rightarrow F_{BP} \triangleq F_{AB}^{-1} \bullet F_{AP} \\
F_{AS} &\rightarrow F_{BS} \triangleq F_{AB}^{-1} \bullet F_{AS} \\
F_{AI} &\rightarrow F_{BI}
\end{align*}
\]

in the formulas used in Questions 1A and 1B, where \( F_{BI} \) is determined by the revised registration process. The registration process becomes

\[
F_{BP,i} \bullet \vec{p}_{PT} = F_{BI} \bullet \vec{p}_{I,i} \quad \text{for } i = 1, 2, 3
\]

\[
F_{AB}^{-1} F_{AP,i} \bullet \vec{p}_{PT} = F_{BI} \bullet \vec{p}_{I,i}
\]

After solving for \( F_{BI} \), the new desired osteotome sensor value at the cut point will be given by

\[
\begin{align*}
F_{BS,0} &= F_{BI} \bullet F_{IC} \bullet F_{ST}^{-1} \\
F_{AB}^{-1} \bullet F_{AS,0} &= F_{BI} \bullet F_{IC} \bullet F_{ST}^{-1} \\
F_{AS,0} &= F_{AB} \bullet F_{BI} \bullet F_{IC} \bullet F_{ST}^{-1}
\end{align*}
\]
As a practical matter, it may be convenient to organize programs so that the “referencing” to a reference body is done in a separate module, so as to create a “virtual tracker”, so that most of the program uses values $F_{BP}$, $F_{BS}$, etc.

**Question 4**

The constraint imposed by the registration step is that the shaft (z-axis) of the probe tip lies along the z-axis of the pin. The image processing software can determine the position and direction of each pin in image coordinates.

$$F_{pin,i} = [R_{pin,i}, \vec{p}_{pin,i}]$$

The position and orientation of the probe tip relative to the pin coordinate system is given by

$$F_{pin-to-probe,i} = (F_{BI} \bullet F_{pin,i})^{-1} \bullet F_{BP,i} \bullet F_{PT}$$

$$= F_{pin,i}^{-1} \bullet F_{BI}^{-1} \bullet F_{BP,i} \bullet F_{PT}$$

The constraint that the axes point in the same direction may be expressed as follows

$$R_{BP,i} \bullet R_{BP} \bullet \vec{z} = R_{BI} \bullet \vec{z}_{pin,i}$$

where $\vec{z}_{pin,i} = R_{pin,i} \bullet \vec{z}$. The constraint that the probe tip must lie along the pin axis may be expressed as
\[ \mathbf{F}_{BP,i} \cdot \mathbf{p}_{PT} = \mathbf{F}_{BI} \cdot \mathbf{p}_{pin,i} + \lambda_i \mathbf{z}_{pin,i} \]

\[ = \mathbf{R}_{BI} \cdot \mathbf{p}_{pin,i} + \mathbf{p}_{BI} + \lambda_i \mathbf{z}_{pin,i} \]

for some unknown real number \( \lambda_i \). The resulting set of orientation and position constraints may be solved for \( \{ \mathbf{R}_{BI}, \mathbf{p}_{BI}, \lambda_1, \lambda_2, \lambda_3 \} \). There are many techniques. Note that the axis direction constraints are sufficient to solve for \( \mathbf{R}_{BI} \), this leaves a simple linear system of 9 equations in 6 unknowns to solve for \( \{ \mathbf{p}_{BI}, \lambda_1, \lambda_2, \lambda_3 \} \).