Streaming verification of graph problems

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We no longer need to do our own computations: we can outsource them!
• Client (verifier) has computationally limited access to the data.
• Server (prover) reads data and has all-powerful access.
• Server must **convince** client that provided answer is correct.
IPs for Muggles [GKR, KRR, others]
- weaker verifiers and provers
- cryptographic assumptions
- verifier TIME key bottleneck
Prior Work

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- Prover is rational, not adversarial
- design a "payment" scheme to convince prover that honesty is optimal
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**Proofs of proximity** [RVW, GR]
- sublinear TIME verifier
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Prover and verifier read the stream
Verifier stores a small amount of information
Prover and verifier interact to determine the answer
Stream of updates $\tau$ of the form $\tau_j = (i, \Delta_{i,j})$

- $i \in [u]$
- $\Delta \in \{+1, -1\}$

Updates can be assembled into a vector

$$a = (a_1, a_2, \ldots, a_u)$$

where $a_i = \sum_j \Delta_{i,j}$
Measuring cost

**Space:**
We would like the verifier to use a working space that is *sublinear* in the input domain size:

\[ s = o(u) \]

**Communication:**
Total communication between the prover and verifier should also be *sublinear* in \( u \):

\[ c = o(u) \]

**Rounds:**
Ideally, total rounds of communication should be small:

\[ r \text{ should be } O(\log u) \text{ or even } O(1). \]

We will describe the cost of a protocol by the pair \((s, c)\)

**Correctness:**
Protocol is randomized:

- If answer is correct, then there exists a proof that convinces verifier with certainty.
- If answer is wrong, then no proof convinces verifier with probability more than \(1/3\)
Prior Work

- Annotated streams [CCM,CCMY,CTM]: Prover helps verifier as stream goes along
- Streaming interactive proofs [CTY]: Introduce the idea of streaming interactive proofs
- Constant-round SIPs [CCMTV] for near neighbors, classification, and median finding, as well as complexity characterization.
- Constant- and log $n$ round SIPs for clustering, shape fitting and eigenvector verification [DTV]
Graph Streams

Graph $G = (V, E)$, $|V| = u$, $|E| = m$ is presented as:

- **Insert-only** stream of edges $e \in E$
- **Dynamic** stream of updates $(e, \Delta)$, $\Delta \in \{+1, -1\}$.

Can’t do anything with $o(u)$ space!

**Semi-streaming model:** allow space $\Omega(u)$ but $o(m)$.

- Connectivity easy in insert-only stream.
- Connectivity easy in dynamic streams (via linear sketches)
- Matchings hard to approximate in dynamic streams
  - Cannot get better than a constant factor approximation using $\tilde{O}(u)$ space [K]
  - Linear sketches require $\Omega(u^{2-o(1)})$ space for constant factor approximation [AKLY]
- If we allow one round of communication ($P \rightarrow V$), then space $\times$ communication is $\Omega(u^2)$ for exact matching [T]
**Our Results**

**Matchings** (all flavors): $O(\log u, \rho + \log u)$ protocols in $\log n$ rounds ($\rho$ is the certificate size). Rounds can be reduced to constant if certificate is large enough.

**TSP** $O(\log n, n \log n)$ protocol for verifying $1.5 + \epsilon$ approximation to TSP (open whether semi-streaming algorithm can do better than 2 even for insert-only streams).

**Triangle Counting** $O(\log n, \log n)$ in $\log n$ rounds (exact).

**Connectivity, Bipartiteness, MST** $(\log n, n \log n)$ protocols.

In all cases, we linearize the graph (via matrix or tensor operations) and do (low-degree) algebraic testing on the resulting vectors.
Some Tools
Lemma (S-Z D-L)

If $p \neq q$ are degree-$d$ polynomials, then

$$\Pr_{r \in_R \mathbb{F}} [p(r) = q(r)] \leq \frac{d}{|\mathbb{F}|}$$

Fix a function $h: \mathbb{Z} \rightarrow \mathbb{Z}$. Set $F(a) = \sum_{i \in [u]} h(a_i)$

Problem (SumCheck)

Verify a claim that $F(a) = K$

Problem formulated in context of interactive proofs.
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Theorem (CTY)

Fix a finite field \( \mathbb{F} \). There is a \( \log u \)-round SIP for SumCheck with cost \((\log u, \deg(h) \log u)\), where \( \deg(h) \) is the degree of a relaxation of \( h \) to \( \mathbb{F} \).

Note that by interpolation, any function \( h \) over a domain of size \( m \) can be written as a polynomial of degree \( m \). Costs are expressed as the number of \textit{words} of \( \mathbb{F} \) needed.
Implications

- If $h(x) = x^2$, we get $F_2$ estimation: $\sum_i a_i^2$
- If $h(x) = 1$ for $x > 0$ and 0 otherwise, we get $F_0$: number of nonzero entries of $a$.
- We can verify $F_0, F_2, F_k, F_{\text{max}}$ exactly using log $n$ space with a streaming verifier.
Implications

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• We can verify \( F_0, F_2, F_k, F_{\text{max}} \) exactly using \( \log n \) space with a streaming verifier.

By comparison with streaming:

• \( \Omega(n) \) space lower bound for an exact streaming algorithm.

• Cannot even approximate \( F_k, k \geq 3 \) in \( o(n^{1-2/k}) \) space streaming.
Let $M = \max_i a_i$. Fix $k \in [M]$. 

$$F_k^{-1}(a) = |\{a_i \mid a_i = k\}|$$

$F_k^{-1}(a)$ is the number of elements with frequency $k$.

**Theorem (Finv)**

There is a SIP to verify a claim that $F_k^{-1}(a) = K$ that has cost $(\log n, M \log n)$ and takes $\log n$ rounds.

Let $h_k(i) = 1$ if $i = k$ and is zero otherwise. Then

$$F_k^{-1}(a) = \sum_i h_k(a_i)$$

and $h$ has degree at most $M$ by interpolation.
Bipartite Maximum Cardinality Matchings

Problem

Given a bipartite graph $G = (A \cup B, E)$, find a set of edges $M \subseteq E$ so that

- each vertex of $A \cup B$ is adjacent to at most one edge of $M$
- $|M|$ is maximized.

Prover has to do two things

- Present a candidate matching
- Convince the verifier that this is optimal

Theorem (König)

In a bipartite graph, size of maximum cardinality matching equal size of minimum vertex cover.

Protocol:

1. V preprocesses the input stream
2. P sends V a matching, and convinces V that it is indeed a matching.
3. P sends V a vertex cover, and convinces V that it is indeed a vertex cover.
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Checking that $M \subset E$
Vector $a$ has one entry for each edge.

1. P and V agree on a canonical ordering of all edges
2. V processes input stream for $F_{-1}^{-1}$ query.
3. P sends back claimed matching $M$ in increasing order. V checks that there are no duplicate edges and decrements $a$ for each edge in $M$.
4. V verifies that $F_{-1}^{-1}(a) = 0$. 
Certifying a Matching I: Subgraph check

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- If $M \subset E$, P passes the test.
- If $M \not\subset E$, then for $e \in M \setminus E$, $a_e = -1$ and so $F_{-1}^{-1}(a) \neq 0$. If $M$ has duplicate entries to inflate the alleged matching, then it will be detected.
Theorem (Multiset Equality, CMT)

Suppose we have streaming updates to two vectors $a, a' \in \mathbb{Z}^u$ such that $\max_i a_i, \max_i a'_i \leq M$. Let $t = \max(M, u)$. Then there is a streaming algorithm using $\log t$ space that outputs 1 if $a = a'$ and outputs 1 with probability $1/t^2$ if $a \neq a'$. 
Certifying a matching II: $M$ is a matching

**Theorem (Multiset Equality, CMT)**

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Now to check if $M$ is a matching:

1. V uses $M$ to construct a stream of updates to the vertices of $G$.
2. V asks P to replay the vertices of $M$ in a canonical order.
3. V verifies that these two sets are identical using Multiset Equality.

Canonical ordering of vertices is needed so that prover cannot cheat by not sending a matching.
A set $S \subseteq V$ is a vertex cover if each edge $e \in E$ is adjacent to some vertex of $S$.

Vector $a$ has one entry for each edge in $E$.

1. $V$ processes data stream for $F_1^{-1}$ query
2. $P$ sends a stream of vertices in $S$ as claimed vertex cover.
3. For each vertex $v \in S$, $V$ simulates the stream of updates $(v, w, -1)$ for all $w \in V$.
4. $V$ verifies at end of stream that $F_1^{-1}(a) = 0$.

If any edge is left uncovered, then its original count is 1 and this is never decremented.
Subgraph Check \((\log n, |M| + \log n)\) via Finv

Matching Check \((\log n, |M| + \log n)\) via MultiSetEquality

Vertex Cover Check \((\log n, |M| + \log n)\) via Finv

- Note that in all invocations of Finv the range of values of \(a_i\) is small.
- Overall protocol takes \(\log n\) rounds.
• Let $w_{ij}$ be the weight of an edge $e = \{i, j\}$.
• Fix (dual) variables $y_v$ and $z_U$, where $U$ is odd-size subset of $V$

**Theorem (Cunningham-Marsh, LP-duality)**

For every integral edge weights $\{w_{ij}\}$, and choices of $y, z$ such that for all $i, j$

$$y_i + y_j + \sum_{\text{odd } U, i, j \in U} z_U \geq w_{ij}$$

we have that

$$c^* \leq \sum_v y_v + \sum_{\text{odd } U} z_U \left\lfloor \frac{1}{2} |U| \right\rfloor$$

And this bound is tight for a laminar family $\{U \mid z_U > 0\}$

• In a laminar family of sets any pair of sets are either disjoint or are nested.
• Therefore a laminar family over a universe of size $u$ is of size at most $u$. 
A few more technical notes

- We can reduce the number of verification rounds to $c = O(1)$ if we allow communication to increase to $n^{1/c}$
- Protocols ignore verifier time: this can also be reduced by increasing the space slightly.
Overview Of Results

Sum check

- MSE
- Finv
- Subset

Verify Matching

Matchings (all variants)

Connectivity

- MST
- Approx TSP
- Triangles
• Graphs are hard to process in a stream: but with a little help, we can solve many graph problems with limited space.

• We don’t understand the full power of SIPs: lower bounds (for constant rounds) are linked to known hard classes like AM.

• There are three canonical hard problems for streaming problems: INDEX, DISJOINTNESS and Boolean Hidden (hyper)Matching. All are easy for SIPs.

• What are candidate hard problems for the SIP model in $\log n$ rounds?
Conclusions

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Thank You!
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