New algorithms for testing monotonicity

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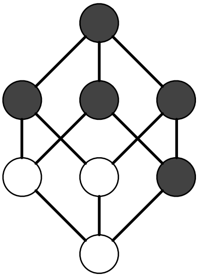
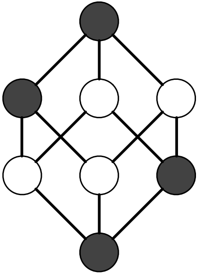
Monotone functions



Definition (Monotone functions; $\mathcal{M}$)

$f : \{0,1\}^n \to \{0,1\}$ is *monotone* if for every $x \preceq y \in \{0,1\}^n$, it satisfies $f(x) \leq f(y)$.

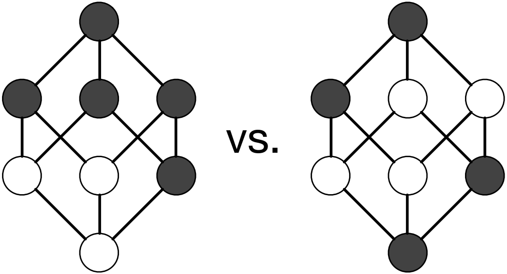
Functions that are far from monotone



Definition (Functions far from monotone; $\overline{\mathcal{M}}_\epsilon$)

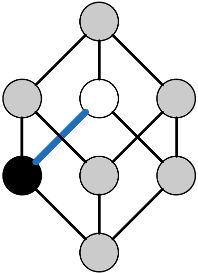
$f : \{0,1\}^n \to \{0,1\}$ is *ϵ-far from monotone* if for every monotone function g, we have $|\{x : f(x) \neq g(x)\}| \geq \epsilon 2^n$.

Testing monotonicity



How many queries does a bounded-error randomized algorithm need to distinguish monotone functions from functions that are ϵ-far from monotone?

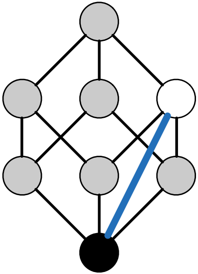
Edge tester



Definition (Goldreich, Goldwasser, Lehman, Ron '98)

The *edge tester* selects edges (x, y) of the hypercube uniformly at random and checks that $f(x) \leq f(y)$.

Pair testers



Definition (Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsky '99)

A *pair tester* selects comparable pairs $x \preceq y \in \{0,1\}^n$ from some distribution and checks that $f(x) \leq f(y)$.

Another view of pair testers

The query complexity of pair testers can also be viewed as the solution to the following optimization problem.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{x \preceq y} \phi_{x,y} & \\
\text{subject to} \quad & \sum_{x \preceq y : f(x) > f(y)} \phi_{x,y} \geq 1 & \forall f \in \overline{\mathcal{M}}_\epsilon \\
& \phi_{x,y} \geq 0 & \forall x \preceq y \in \{0,1\}^n
\end{aligned}
$$

A different optimization problem

$$\text{minimize} \quad \max_{f \in \mathcal{M} \cup \overline{\mathcal{M}}_\epsilon} \sum_x \left(\sum_{y \succeq x} \phi_{x,y}(f) \right)^2$$

$$\text{subject to} \quad \sum_{x : f(x) \neq g(x)} \left(\sum_{y \succeq x} \phi_{x,y}(f) \cdot \phi_{x,y}(g) \right) = 1 \quad \forall f \in \mathcal{M}, g \in \overline{\mathcal{M}}_\epsilon.$$

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Corollary (to the Dual adversary bound Theorem)

Every feasible solution to this problem gives an upper bound on the quantum query complexity for testing monotonicity.

The dual adversary bound

Theorem (Dual adversary bound)

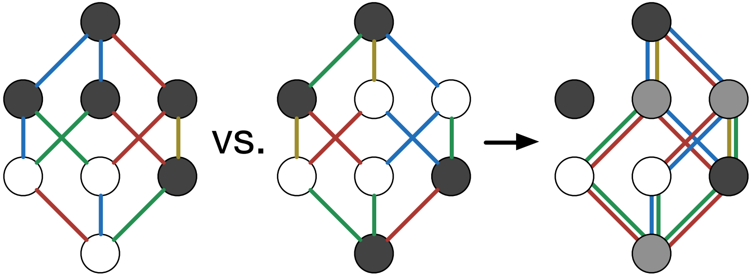
The quantum query complexity for distinguishing $\mathcal{X}$ and $\mathcal{Y}$ is the solution to the optimization problem

$$\begin{aligned} &\textit{minimize} && \max_{f \in \mathcal{X} \cup \mathcal{Y}} \sum_{x} X_x[f,f] \\ &\textit{subject to} && \sum_{x : f(x) \neq g(x)} X_x[f,g] = 1 \quad \forall f \in \mathcal{X}, g \in \mathcal{Y} \\ & && X_x \succeq 0 \quad \forall x \in \{0,1\}^n \end{aligned}$$

Simplifying the optimization problem

$$\text{minimize} \quad \max_{f \in \mathcal{M} \cup \overline{\mathcal{M}}_\epsilon} \sum_x \left(\sum_{j \in [n]} \phi_{x,j}(f) \right)^2$$

$$\text{s.t.} \quad \sum_{x: f(x) \neq g(x)} \sum_{j \in [n]} \phi_{x,j}(f) \cdot \phi_{x,j}(g) = 1 \quad \forall f \in \mathcal{M}, g \in \overline{\mathcal{M}}_\epsilon.$$



First quantum monotonicity tester

For $f \in \mathcal{M}$, define

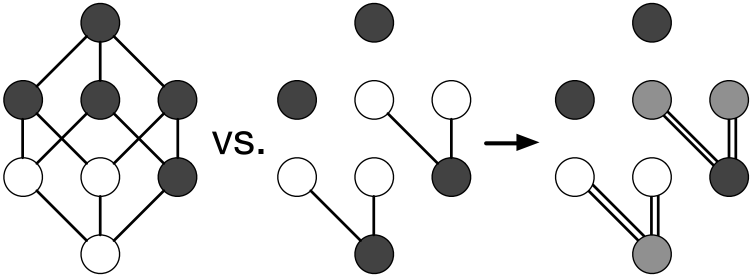
$$\phi_{x,j}(f) = \begin{cases} 1/L & \text{if } x_j = 0 \text{ and } f(x) = 0 \\ & \text{or } x_j = 1 \text{ and } f(x) = f(x^{\oplus j}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \overline{\mathcal{M}}_\epsilon$, define

$$\phi_{x,j}(g) = \begin{cases} L/|E_g| & \text{if } (x, x^{\oplus j}) \in E_g \\ 0 & \text{otherwise} \end{cases}$$

where E_g is the set of edges of the hypercube on which g is anti-monotone and L is a constant to be fixed later.

First quantum tester: Correctness



$$\sum_{x:f(x)\neq g(x)} \sum_{j\in[n]} \phi_{x,j}(f)\cdot\phi_{x,j}(g) = |E_g|\cdot(\frac{1}{L}\cdot\frac{L}{|E_g|}) = 1.$$

First quantum tester: Complexity I

For $f \in \mathcal{M}$, the objective value of the optimization is

$$\sum_x \left(\sum_{j \in [n]} \phi_{x,j}(f) \right)^2 = \frac{n2^n}{L^2}$$

And for $g \in \overline{\mathcal{M}}_\epsilon$, it is

$$\sum_x \left(\sum_{j \in [n]} \phi_{x,j}(g) \right)^2 = 2|E_g| \frac{L}{|E_g|^2} = \frac{2L}{|E_g|}.$$

First quantum tester: Complexity II

When $L = \sqrt{n\epsilon} \cdot 2^{n-1}$, the objective value of the optimization problem is

$$\max\left\{\sqrt{n/\epsilon}, \max_{g \in \overline{\mathcal{M}}_\epsilon} \frac{2^n \sqrt{n\epsilon}}{|E_g|}\right\}.$$

First quantum tester: Complexity II

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Lemma (Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky '00)

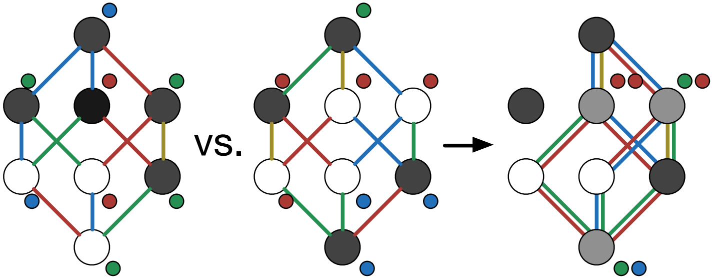
For every $g \in \overline{\mathcal{M}}_\epsilon$, $|E_g| \geq \epsilon 2^n$.

So the quantum query complexity of the first tester is $\sqrt{n/\epsilon}$.

A more flexible optimization problem

$$\text{min.} \quad \max_{f \in \mathcal{M} \cup \overline{\mathcal{M}}_\epsilon} \sum_x \left(\psi_x(f) + \sum_{j \in [n]} \phi_{x,j}(f) \right)^2$$

$$\text{s.t.} \quad \sum_{x : f(x) \neq g(x)} \left(\psi_x(f) \cdot \psi_x(g) + \sum_{j \in [n]} \phi_{x,j}(f) \cdot \phi_{x,j}(g) \right) = 1 \quad \forall \ldots$$



Second quantum monotonicity tester

Theorem (Belovs, B. '15)

There is a feasible solution to this optimization problem with objective value

$$\frac{2^n\sqrt{\epsilon}}{\log n|E_g|}\left(\frac{\Delta(G_g)}{n^{1/4}} + n^{1/4}\right)$$

where G_g *is any subgraph of the* $(1,0)$*-graph of* g, $\Delta(G_g)$ *is its maximum degree, and* E_g *is the set of non-monotone edges in* G_g*.*

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Theorem (Khot, Minzer, Safra '15)

For every $g \in \overline{\mathcal{M}}_\epsilon$, there exists a such a subgraph G_g that satisfies

$$|E_g| = \Omega\left(\frac{\epsilon 2^n\sqrt{\Delta(G_g)}}{\log^2 n}\right).$$

Conclusions

- We can test monotonicity with $\tilde{O}(n^{1/4}/\sqrt{\epsilon})$ quantum queries.
- The design of quantum testers can be done by considering natural optimization problems.
- The analysis of quantum monotonicity testers uncovers the key inequalities that are also required to analyze classical monotonicity testers.
- Are there other property testing problems where considering quantum testers may yield insights on promising directions?

Thank you!

For all the details, see

A. Belovs and E.B. Quantum Algorithm for Monotonicity Testing on the Hypercube. *Theory of Computing* 11(16), 2015.