Feasibility Pump for Mixed Integer Nonlinear Programs[1]

Presenter: Svitlana Volkova

[1] by Pierre Bonami, Gerard Cornuejols, Andrea Lodi and Francois Margot

Mixed Integer Linear or Nonlinear Programs (MILP/MINLP)

- Optimize objective function:

$$\min_{f:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}} f(x,y)$$

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$$\min_{f:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}} f(x,y)$$

- Subject to constraints:

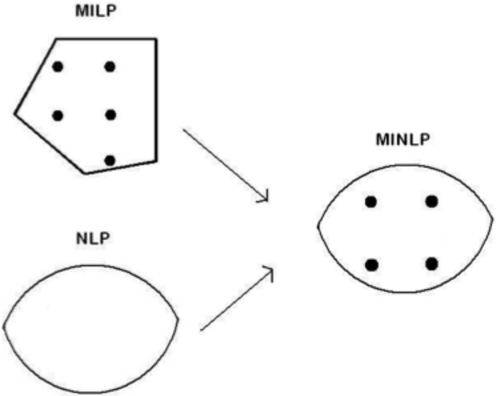
$$g(x,y) \leq b, g : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$$

$$x \in \mathbb{Z}^{n_1}$$

$$y \in \mathbb{R}^{n_2}$$

The relationship between MINLP, NLP and MILP [2]

- If f and g are linear functions ⇒ **MILP**.
- If all variables are continuous ⇒ **NLP**.



[2] A Tutorial on Mixed-Integer Non-Linear Programming by A. Letchfold, 2010

Mixed Integer Nonlinear Programs Applications[3]

- best-subset multiple linear regression
- various portfolio optimization problems
- robust versions of MILPs
- machine scheduling problems with min-variance objective
- inventory-routing problems design of networks for electricity transmission
- design of chemical processes
- scheduling gas- or coal-fired power stations

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Methods for Solving MINLP

- Branch and bound method (BB) (Gupta and Ravindran, 1985; Nabar and Schrage, 1991; Borchers and Mitchell, 1994; Stubbs and Mehrotra, 1996; Leyffer, 1998).
- Generalized Benders Decomposition (GBD) (Geoffrion, 1972).
- Outer-Approximation (OA) (Duran and Grossmann, 1986; Yuan et al., 1988; Fletcher and Leyffer, 1994).
- LP/NLP based branch and bound (Quesada and Grossmann, 1992).
- Extended Cutting Plane Method (ECP) (Westerlund and Pettersson, 1995).

A Feasibility Pump for Mixed Integer Linear Program (MILP)

- Generate a sequence of integer infeasible points $(\bar{x}^0, \bar{y}^0), \ldots, (\bar{x}^k, \bar{y}^k)$ that satisfy the continuous relaxation.

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- Sequence of integer feasible points $(\hat{x}^1, \hat{y}^1), \ldots, (\hat{x}^{k+1}, \hat{y}^{k+1})$ that do not necessarily satisfy all constraints ($\hat{x}^{i+1}$ is a componentwise rounding of $\bar{x}_i$; $\hat{y}^{i+1} = \bar{y}^i$).

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- Sequence $(\bar{x}^i, \bar{y}^i)$ is generated by solving LP whose objective is to minimize the distance between x to $\hat{x}^i$ according to L_1 norm.

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- Sequence $(\bar{x}^i, \bar{y}^i)$ is generated by solving LP whose objective is to minimize the distance between x to $\hat{x}^i$ according to L_1 norm.
- Two sequences has the property: at each iteration the distance between $\bar{x}^i$ to $\hat{x}^i$ is non-increasing (the procedure may cycle so need to use random restarts).

A Feasibility Pump for Mixed Integer Nonlinear Program (MINLP)

- Construct two sequences:

$$s_1 = (\bar{x}^0, \bar{y}^0), \ldots, (\bar{x}^k, \bar{y}^k)$$

$$s_2 = (\hat{x}^1, \hat{y}^1), \ldots, (\hat{x}^{k+1}, \hat{y}^{k+1})$$

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- that satisfy the properties:
  - points $(\bar{x}^i, \bar{y}^i)$ satisfy $g(\bar{x}^i, \bar{y}^i) \leq b$, but $\bar{x}^i$ is "not necessarily" in $\mathbb{Z}^{n_1}$
  - points $(\hat{x}^i, \hat{y}^i)$ **do not** satisfy $g(\hat{x}^i, \hat{y}^i) \leq b$, but $\hat{x}^i \in \mathbb{Z}^{n_1}$

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- $(\bar{x}^i, \bar{y}^i)$ is generated by solving NLPs
- $(\hat{x}^i, \hat{y}^i)$ is generated by solving MILPs

Feasibility Pump When the Functions g_j are Convex

- Construct the sequence $(\hat{x}^1, \hat{y}^1), \ldots, (\hat{x}^{k+1}, \hat{y}^{k+1})$ by Outer Approximation of the region $g(x, y) \leq b$. It linearizes the constraints of the continuous relaxation of MINLP to build a mixed integer linear relaxation of MINLP.

Continuous Relaxation of MINLP

- $(\bar{x}, \bar{y})$ is a feasible solution and g_j is convex, so the constraints are valid:

$$g_j(\bar{x}, \bar{y}) + \nabla g_j(\bar{x}, \bar{y})^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right) \leq b_j, j = 1, \ldots, m$$

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- Therefore, given any set of points $(\bar{x}^0, \bar{y}^0), \ldots, (\bar{x}^{i-1}, \bar{y}^{i-1})$ can build a relaxation of the feasible set of MINLP:

$$\begin{cases} g(\bar{x}^k, \bar{y}^k) + J_g(\bar{x}^k, \bar{y}^k)^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x}^k \\ \bar{y}^k \end{pmatrix} \right) \leq b, \forall k = 0, \ldots, i-1 \\ x \in \mathbb{Z}^{n_1} \\ y \in \mathbb{R}^{n_2} \end{cases}$$

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- Solution: $(\hat{x}^i, \hat{y}^i)$.

The Basic Feasibility Pump: Step 1

Step 1: Feasibility Outer Approximation $(FOA)^i$

- Choose $(\bar{x}^0, \bar{y}^0)$ to be an optimal solution of the continuous relaxation of MINLP.

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Step 1: Feasibility Outer Approximation $(FOA)^i$

- Choose $(\bar{x}^0, \bar{y}^0)$ to be an optimal solution of the continuous relaxation of MINLP.
- For $i \geq 1$ find a point $(\hat{x}^i, \hat{y}^i)$ that solves:

$$\begin{cases} min\|(x-\bar{x}^{i-1})\|_1 \\ s.t. \\ g(\bar{x}^k,\bar{y}^k) + J_g(\bar{x}^k,\bar{y}^k)^T\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x}^k \\ \bar{y}^k \end{pmatrix}\right) \leq b, \forall k = 0,\dots,i-1 \\ x \in \mathbb{Z}^{n_1} \\ y \in \mathbb{R}^{n_2} \end{cases}$$

The Basic Feasibility Pump: Step 2

Step 2: Feasibility Pump - NonLinear Programs $(FP - NLP)^i$

- Compute $(\bar{x}^i, \bar{y}^i)$ by solving NLP:

$$\begin{cases} min\|(x - \hat{x}^i)\|_2 \\ s.t. \\ g(x, y) \leq b \\ x \in \mathbb{R}^{n_1} \\ y \in \mathbb{R}^{n_2} \end{cases}$$

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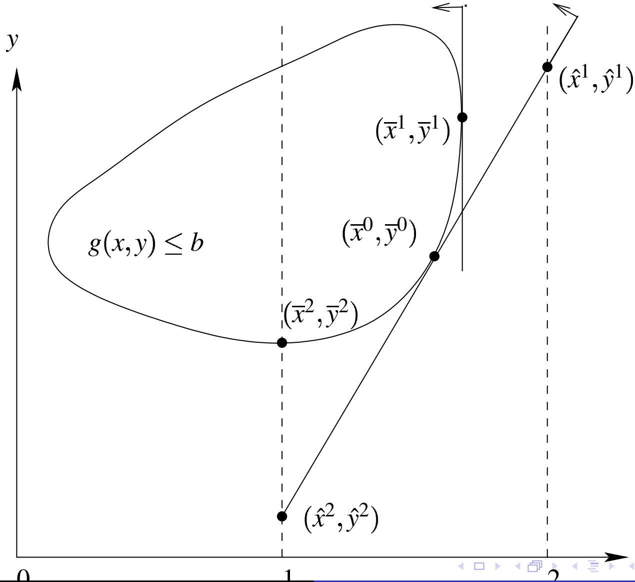
- The basic FP iterates between solving $(FOA)^i$ and $(FP-NLP)^i$ until either a feasible solution of MINLP is found or $(FOA)^i$ becomes infeasible.

Illustration of The Feasibility Pump



Enhanced Feasibility Pump $(SFOA)^i$

- At iteration $k > 0$ we have a point $(\hat{x}^k, \hat{y}^k)$ outside the convex region $g(x, y) \leq b$ and a point $(\bar{x}^k, \bar{y}^k)$ on its boundary that minimizes $\|(x - \bar{x}^k)\|_2$. Then $(\bar{x}^k - \hat{x}^k)^T(x - \bar{x}^k) \geq 0$ is valid for MINLP.

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- Add additional constraint:

$$
\begin{cases}
min\|(x - \bar{x}^{i-1})\|_1 \\
s.t. \\
g(\bar{x}^k, \bar{y}^k) + J_g(\bar{x}^k, \bar{y}^k)^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x}^k \\ \bar{y}^k \end{pmatrix} \right) \leq b, \forall k = 0, \ldots, i-1 \\
(\bar{x}^k - \hat{x}^k)^T(x - \bar{x}^k) \geq 0, \forall k = 1, \ldots, i-1 \\
x \in \mathbb{Z}^{n_1} \\
y \in \mathbb{R}^{n_2}
\end{cases}
$$

Enhanced Feasibility Pump $(SFOA)^i$

- In general, FP is a heuristic, but when the region

$$S := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : g(x, y) \leq b\}$$

is convex, the enhanced FP is an exact algorithm.

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$$S := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : g(x, y) \le b\}$$

  is convex, the enhanced FP is an exact algorithm.
- When the region $g(x, y) \le b$ is convex and the integer variables x are bounded, the enhanced FP either finds a feasible solution or proves that none exists.

Feasibility Pump Algorithm

1: i = 0;
2: initialize $(\bar{x}^0, \bar{y}^0)$ and $(\hat{x}^0, \hat{y}^0)$);
3: while ($(\bar{x}^i, \bar{y}^i) \neq (\hat{x}^i, \hat{y}^i)$ and CPU time < limit) do
4:     increase i;
5:     solve $(FOA^i)/(SFOA^i)$ (minimize distance to $(\bar{x}^i, \bar{y}^i)$
      s.t. $x \in \mathbb{Z}^{n_1}, y \in \mathbb{R}^{n_2}$) to yield $(\hat{x}^i, \hat{y}^i)$;
6:     solve $(FP - NLP)^i$ (minimize distance to $(\hat{x}^{i-1}, \hat{y}^{i-1})$
      s.t. $x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}$) to yield $(\bar{x}^i, \bar{y}^i)$;
7: end while.

Feasibility Pump When the Region $g(x, y) \leq b$ is convex

- If $g(x, y) \leq b$ is convex but some of the functions are nonconvex (*e.g.*, g_j), then constraint may cut off the part of the feasible region **unless** $(\bar{x}, \bar{y})$ satisfies the constraint $g_j(x, y) \leq b_j$ with equality $g_j(x, y) = b_j$.

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- The constraint is valid for any point $(\bar{x}, \bar{y})$ when g_j is convex:

$$g_j(\bar{x},\bar{y}) + \nabla g_j(\bar{x},\bar{y})^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right) \le b_j, \forall j \in I(\bar{x},\bar{y}),$$

where $I(\bar{x},\bar{y}) = \{j : \text{either } g_j \text{ is convex or } g_j(\bar{x},\bar{y}) = b_j\} \subseteq \{1, \dots, m\}$.

Feasibility Pump $(FP-OA)^i$ When the Region $g(x,y) \le b$ is Convex

$$
\begin{cases}
min\|(x-\bar{x}^{i-1})\|_1 \\
s.t. \\
g_j(\bar{x}^k,\bar{y}^k) + \nabla g_j(\bar{x}^k,\bar{y}^k)^T \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \bar{x}^k \\ \bar{y}^k \end{pmatrix} \right) \le b_j, \\
\qquad \forall j \in I(\bar{x},\bar{y}), \forall k = 0,\dots,i-1 \\
(\bar{x}^k - \hat{x}^k)^T (x - \bar{x}^k) \ge 0, \forall k = 1,\dots,i-1 \\
x \in \mathbb{Z}^{n_1} \\
y \in \mathbb{R}^{n_2}
\end{cases}
$$

- Constraints $(FP-OA)^i$ give a valid outer approximation of MINLP given the region $g(x,y) \le b$ is convex.

Enhanced Feasibility Pump $(FP - OA)^i$ When the Region $g(x, y) \leq b$ is Convex

- Starts with feasible solution $(\bar{x}^0, \bar{y}^0)$ of the continuous relaxation of MILNP.

Enhanced Feasibility Pump $(FP - OA)^i$ When the Region $g(x, y) \leq b$ is Convex

- Starts with feasible solution $(\bar{x}^0, \bar{y}^0)$ of the continuous relaxation of MILNP.
- Iterates solving $(FP - OA)^i$ and $(FP - NLP)^i$ for $i \geq 1$ until either a feasible solution of MINLP is found or $(FP - OA)^i$ becomes infeasible.

Basic Feasibility Pump Convergence

- If linearly independent constraint qualification holds at each point $(\bar{x}^i, \bar{y}^i)$, then the basic FP cannot cycle.
- Constraint qualification holds if the vectors $\nabla g_j(\bar{x}^i, \bar{y}^i)$ are linearly independent for all $i \in J$, where $J \subseteq \{1, \cdots, m\}$ is a set of indices for for which $g_j(x, y) \leq b_j$ is satisfied with equality by $(\bar{x}, \bar{y})$.

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Theorem

In the basic FP, let $(\hat{x}^i, \hat{y}^i)$ be an optimal solution of $(FOA)^i$ and $(\bar{x}^i, \bar{y}^i)$ an optimal solution of $(FP - NLP)^i$. If the constraint qualification for $(FP - NLP^i)$ holds at $(\bar{x}^i, \bar{y}^i)$, then $\bar{x}^i \neq \bar{x}^k$ for all $k = 0, \ldots, m$.

Enhanced Feasibility Pump Convergence

Theorem

The enhanced Feasibility Pump cannot cycle. If the integer variables x are bounded, the enhanced FP terminates in a finite number of iterations. If, in addition, the region $g(x,y) \leq b$ is convex, the enhanced FP is an exact algorithm: either it finds a feasible solution of MINLP if one exists, or it proves that none exists.

Conclusions and Opened Research Questions

Fact

MINLPs are not only NP-hard, they are worse than NP-hard!

- MINLP has many applications
- MINLP are much more difficult to solve than NLP and MILP
- Several methods are available for convex case

- Nonconvex problems
- Need more representative real-world test problems