Augmented Lagrangian Method

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General Constrained Optimization Problem

- Optimize nonlinear objective function

$$\min_{x \in \Re^n} f(x)$$

- Subject to nonlinear constraints:

$$c(x) = 0$$

Sequential Quadratic Programming (SQP)

- **Step Generation:** Apply Newton method to solve 1^{st} order optimality conditions:

$$\min g(x_k)^T \Delta x + \frac{1}{2} \Delta x^T H(x_k, \lambda_k) \Delta x$$
$$\text{s.t. } c(x_k) + J(x_k) \Delta x = 0$$

Sequential Quadratic Programming (SQP)

- **Step Acceptance:** Merit function balances the conflicting goals of reducing the objective function $f(x)$ and the constraint violation $\|c(x)\|$, e.g., L_1 merit function:

$$\Phi(x, \rho) = f(x) + \frac{\rho}{2}\|c(x)\|_1$$

SQP Characteristics

- Not a feasible method
- Solving QP at each iteration is expensive in large scale constrained optimization
- Dependent on rapid algorithms for solving quadratic programs
- Only guarantee to find a local solution

Penalty Methods

- Main idea: solving a constrained optimization problem by solving a sequence of unconstrained optimization problems
- Another type of merit function - **quadratic penalty**

$$P(x, \rho) = f(x) + \frac{\rho}{2}\|c(x)\|_2^2$$

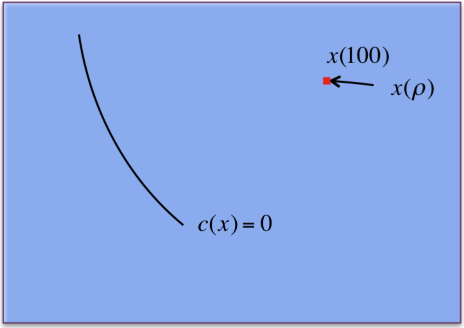
$$x(\rho) = \arg\min_{x \in \Re^n} P(x, \rho)$$

- The solution to original optimization problem:

$$\lim_{\rho \to \infty} x(\rho) = x^*$$

Limitations of Penalty Methods

- $\lim_{\rho \to \infty} x(\rho) = x^*$ is not computationally reasonable



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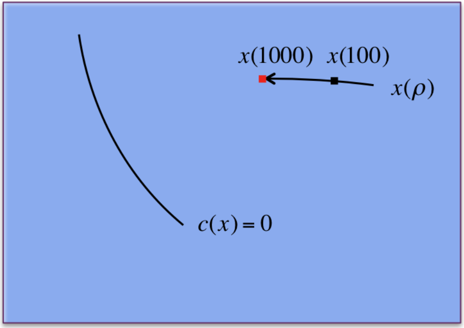

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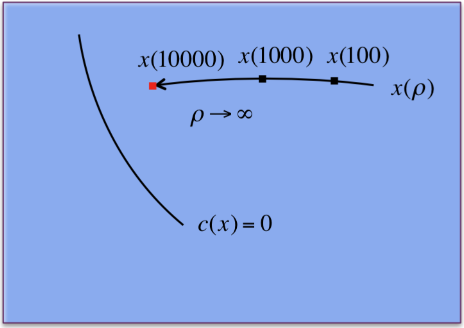

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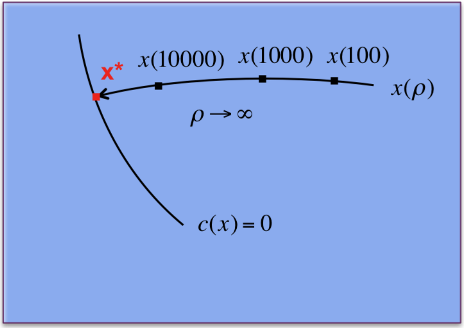
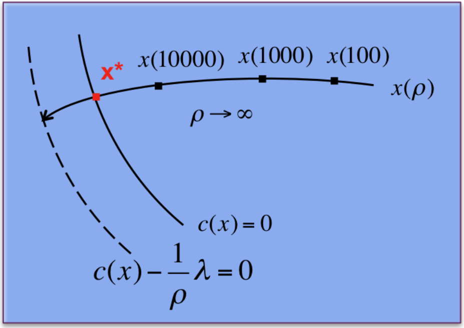


Introducing Augmented Lagrangian Method

- $\lim_{\rho \to \infty} x(\rho) = x^*$ is not computationally reasonable



- If we relax $c(x = 0)$ by some value *e.g.*, $\frac{1}{\rho}\lambda$ then we can achieve x^* with finite ρ

Problem Reformulation

- We reformulate the original problem **NLP**

$$\min_{x \in \Re^n} f(x)$$
$$\text{s.t. } c(x) = 0$$

- into a new problem **PNLP**:

$$\min_{x \in \Re^n} f(x)$$
$$\text{s.t. } c(x) - \frac{1}{\rho}\lambda = 0$$

Introducing Augmented Lagrangian

- Penalty Method:

$$x(\rho) = \arg\min_{x \in \Re^n} f(x) + \frac{\rho}{2}\|c(x)\|_2^2$$

- Augmented Lagrangian:

$$x(\rho, \lambda) = \arg\min_{x \in \Re^n} f(x) + \frac{\rho}{2}\|c(x) - \frac{1}{\rho}\lambda\|_2^2 =$$

$$= \arg\min_{x \in \Re^n} f(x) + \frac{\rho}{2}\|c(x)\|^2 - c(x)^T\lambda$$

$$x(\rho, \lambda) \doteq \arg\min_{x \in \Re^n} \underbrace{f(x) - c(x)^T\lambda + \frac{\rho}{2}\|c(x)\|_2^2}_{\mathcal{L}_A(x,\rho,\lambda)}$$

- optimal shift = optimal multiplier

1^{st} Order Optimality Conditions for Original Optimization Problem

- Original problem

$$\min_{x \in \Re^n} f(x)$$
$$\text{s.t. } c(x) = 0$$

- 1^{st} order optimality conditions for original problem
  - $c(x) = 0$
  - $\nabla f(x) - \nabla c(x)^T \lambda = 0 \Rightarrow g(x) - J(x)^T \lambda = 0$

1^{st} Order Optimality Conditions for Modified Optimization Problem

- Modified problem

$$\min_{x \in \Re^n} f(x)$$
$$\text{s.t. } c(x) - \frac{1}{\rho}\lambda = 0$$
$$L(x, \rho, \lambda) = f(x) - c(x)^T \lambda + \frac{\rho}{2} \|c(x)\|_2^2$$

- 1^{st} order optimality conditions for modified problem:

$$c(x) - \frac{1}{p}\lambda = 0$$
$$\nabla L(x, \rho, \lambda) = \nabla f(x) - \nabla c(x)^T \lambda + \frac{\rho}{2} \nabla \|c(x)\|_2^2 =$$
$$= g(x) - J(x)^T \lambda + \rho J(x)^T c(x)$$
$$g(x) - J(x)^T (\lambda - \rho c(x)) = 0$$

Augmented Lagrangian vs. SQP

- SQP generates a step and then pairs it with a merit function
- AL similarly to penalty methods directly minimizes penalty function

Extend Augmented Lagrangian Method to Inequality Constraints

- Inequality constraints:

$$\min_{x,s \in \Re^n} f(x)$$
$$\text{s.t. } c(x) \geq 0$$

- Use slack variables

$$\min_{x,s \in \Re^n} f(x)$$
$$\text{s.t. } c(x) - s = 0$$
$$s \geq 0$$

- solving a sequence of bounded constrained problems