Potential Functions

Configuration Space

- A key concept for motion planning is a **configuration**:

  a <u>complete</u> specification of the position of every point in the system

- A simple example: a robot that translates but does not rotate in the plane:
  - what is a sufficient representation of its configuration?

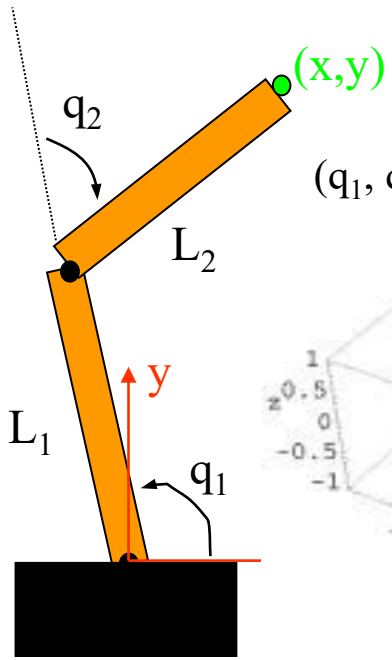
The space of all configurations is the configuration space or C-space.

C-space formalism:
Lozano-Perez '79

Three Fundamental Issues with C-space

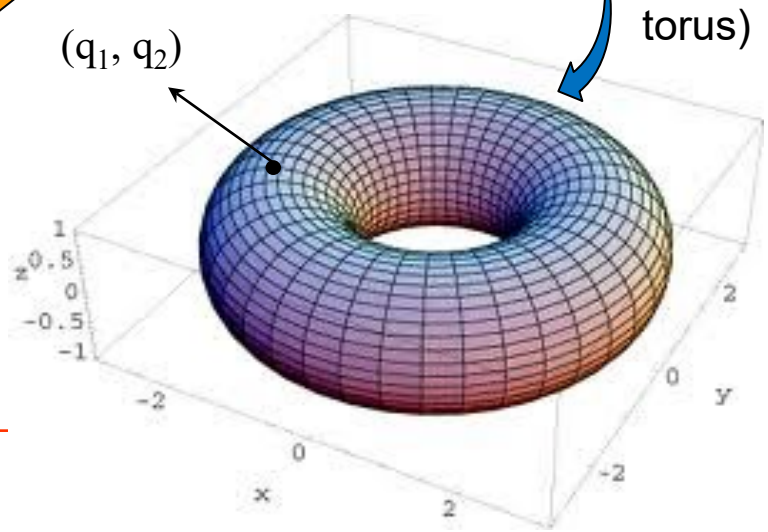
- How do we describe configurations?
  - What is the fundamental space of configurations for a mechanisms?
  - How we relate the configuration space to the workspace? Kinematics
- How are configurations related to controllable inputs to the system?
  - Holonomic vs. Non-holonomic systems
  - Effect of dynamics
- How are obstacles represented in C-space
  - Explicit mapping via Inverse kinematics
  - Implicit in algorithm

Configuration Space



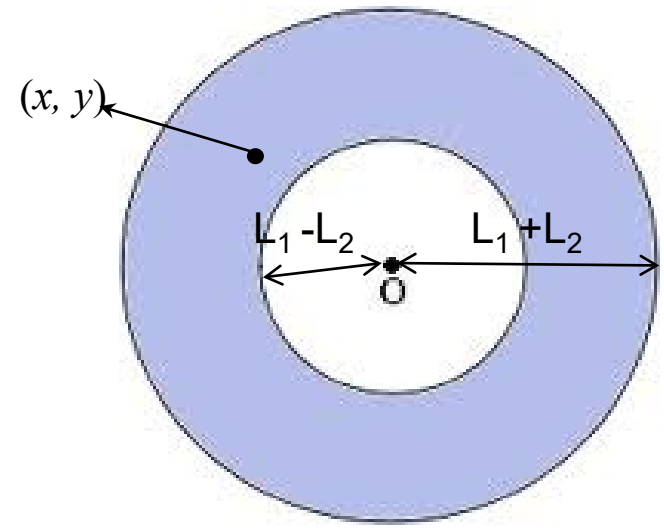
$S^1 \times S^1$

Cartesian product of two circles (2D torus)



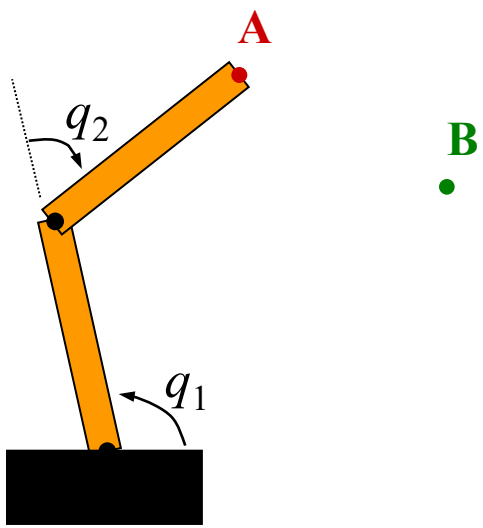
Topology of Configuration Space

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_1\cos(q_1) \\ L_1\cos(q_1) \end{bmatrix} + \begin{bmatrix} L_1\cos(q_1+q_2) \\ L_1\cos(q_1+q_2) \end{bmatrix}$$

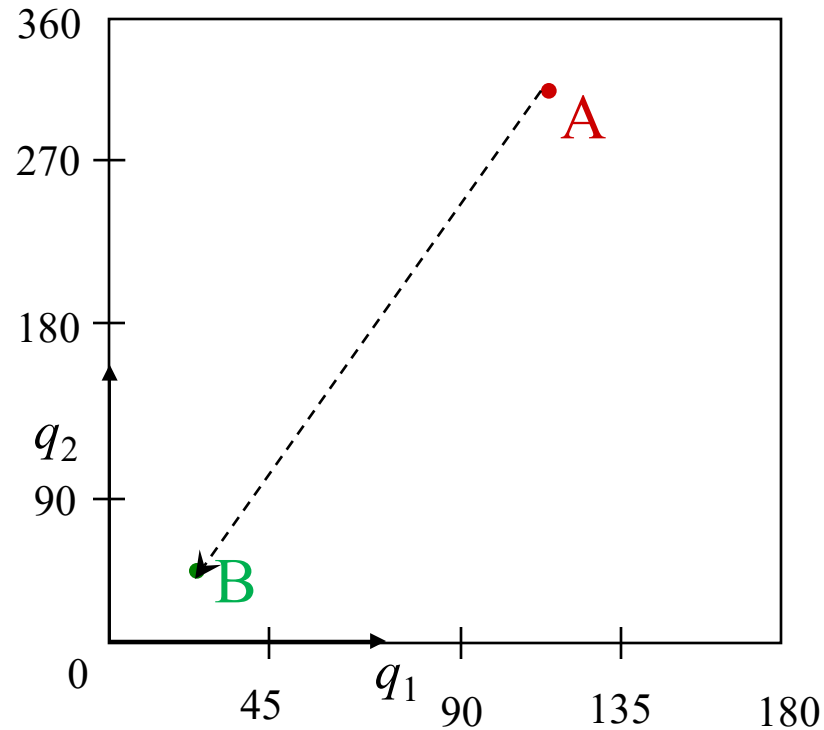


The Workspace

Configuration Space



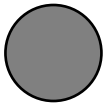
Where do we put **B**?

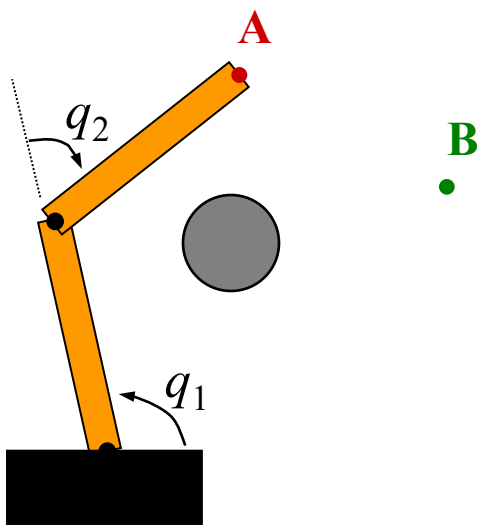


Torus

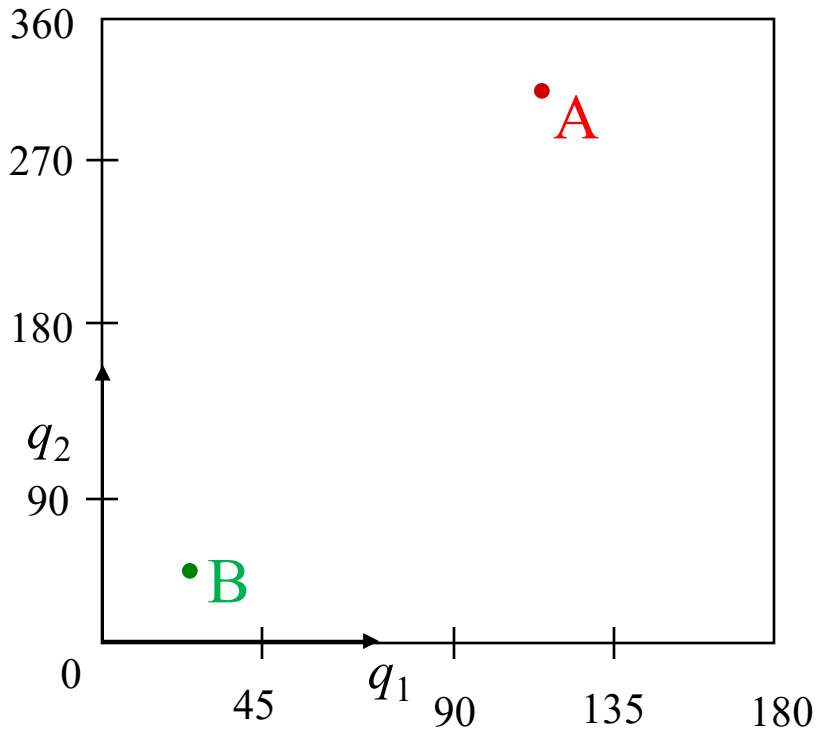
(wraps horizontally and vertically)

Configuration Space

Where do we put  ?



An obstacle in the robot's workspace

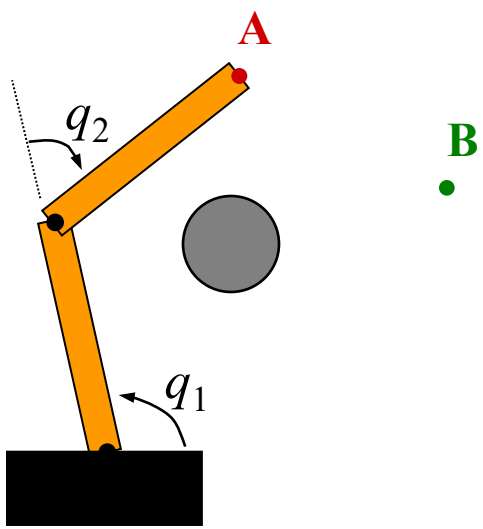


Torus

(wraps horizontally and vertically)

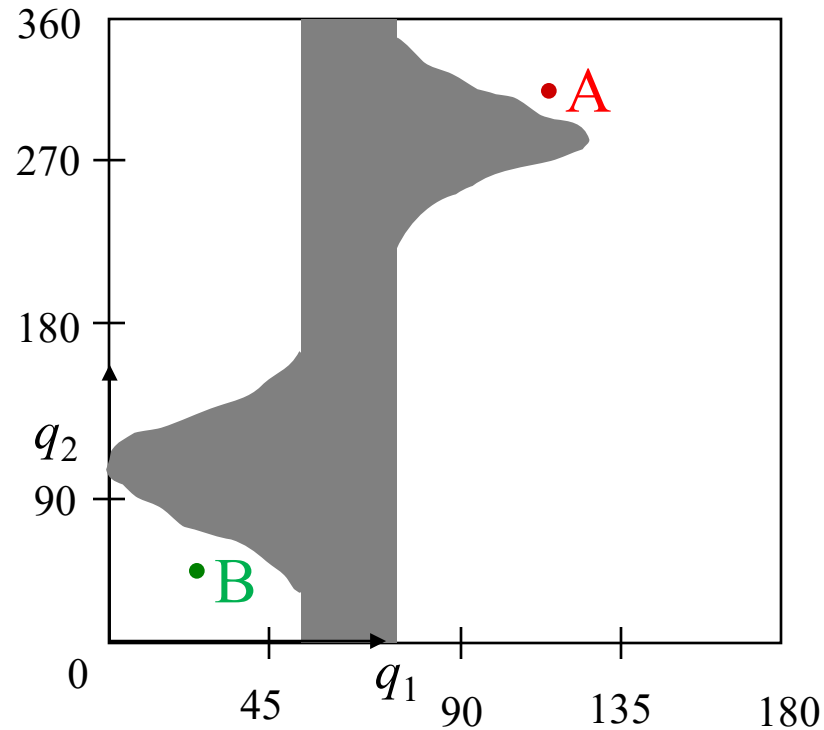
Configuration Space

Reference *configuration*



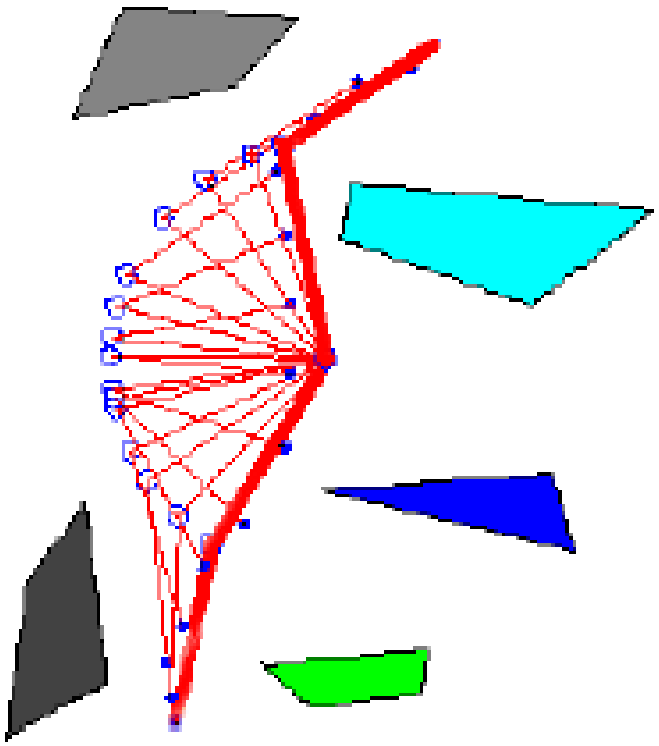
An obstacle in the robot's workspace

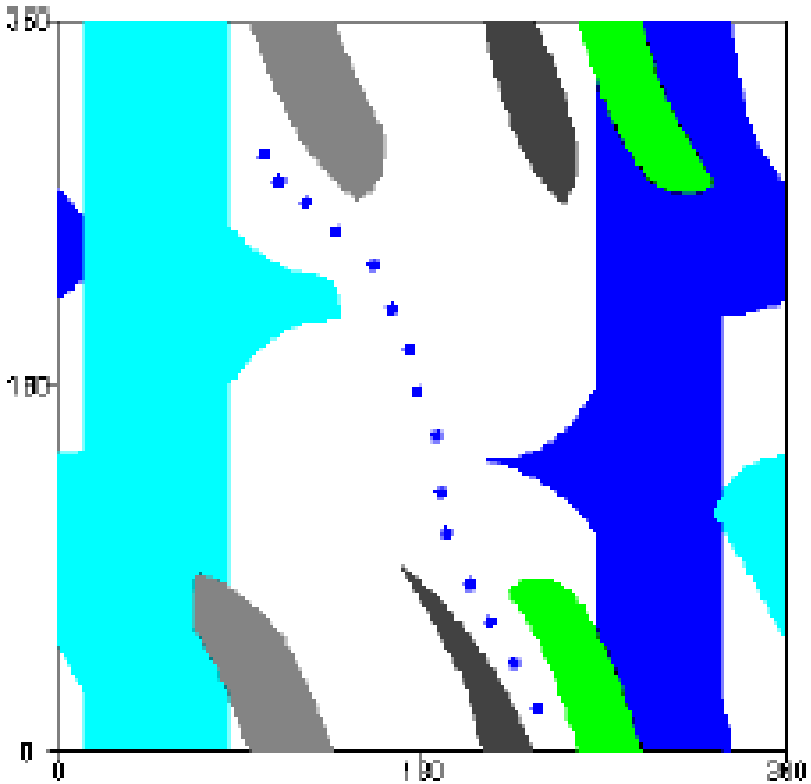
How do we get from **A** to **B** ?



The C-space representation of this obstacle…

Two Link Path



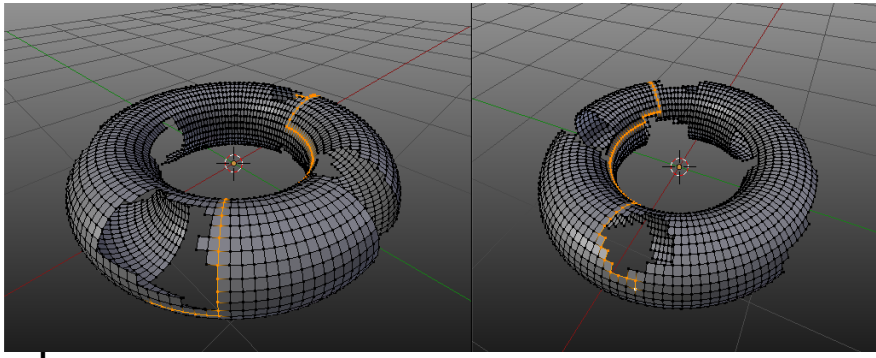


Thanks to Ken Goldberg

Some Other Examples of C-Space

- A rotating bar fixed at a point
  - what is its C-space?
  - what is its workspace (the set of points it can reach?)

- A rotating bar that translates along the rotation axis
  - what is its C-space?
  - what is its workspace

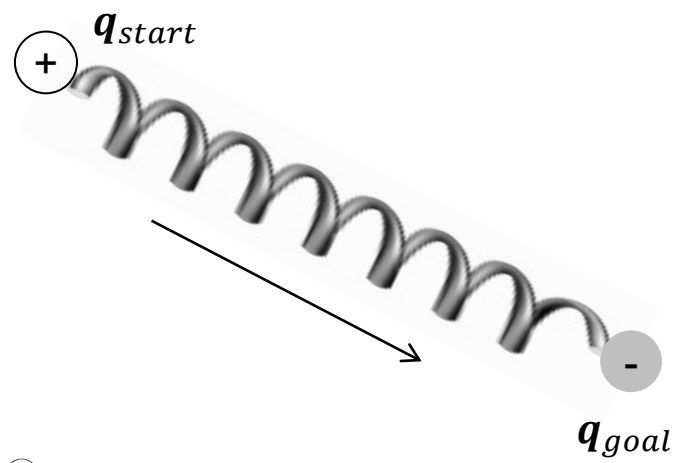


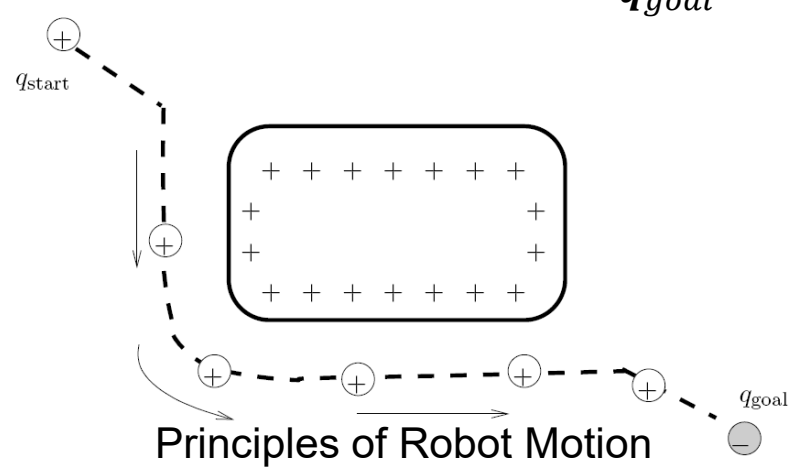
- A two-link manipulator
  - what is its C-space?
  - what is its workspace?
  - Suppose there are joint limits, does this change the C-space?
  - The workspace?

Potential Field: The Basic Idea

A really simple idea:

- Suppose the configuration goal is a point $\boldsymbol{q}_{goal} \in \mathbb{R}^2$
- Suppose the robot is at configuration $\boldsymbol{q}_{start} \in \mathbb{R}^2$
- Think of a "spring" drawing the robot toward the goal and away from obstacles:
- Can also think of using same/opposite electric charges to repeal/attract

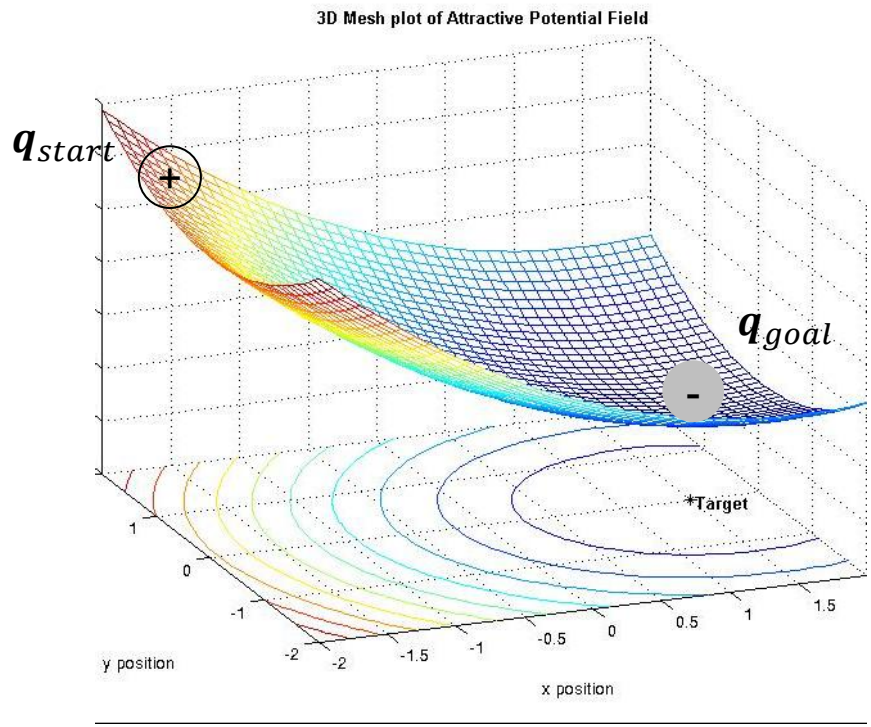




Principles of Robot Motion

Another Idea

- Think of the goal as the bottom of a bowl
- The robot is a ball on the rim of the bowl
- What will happen?



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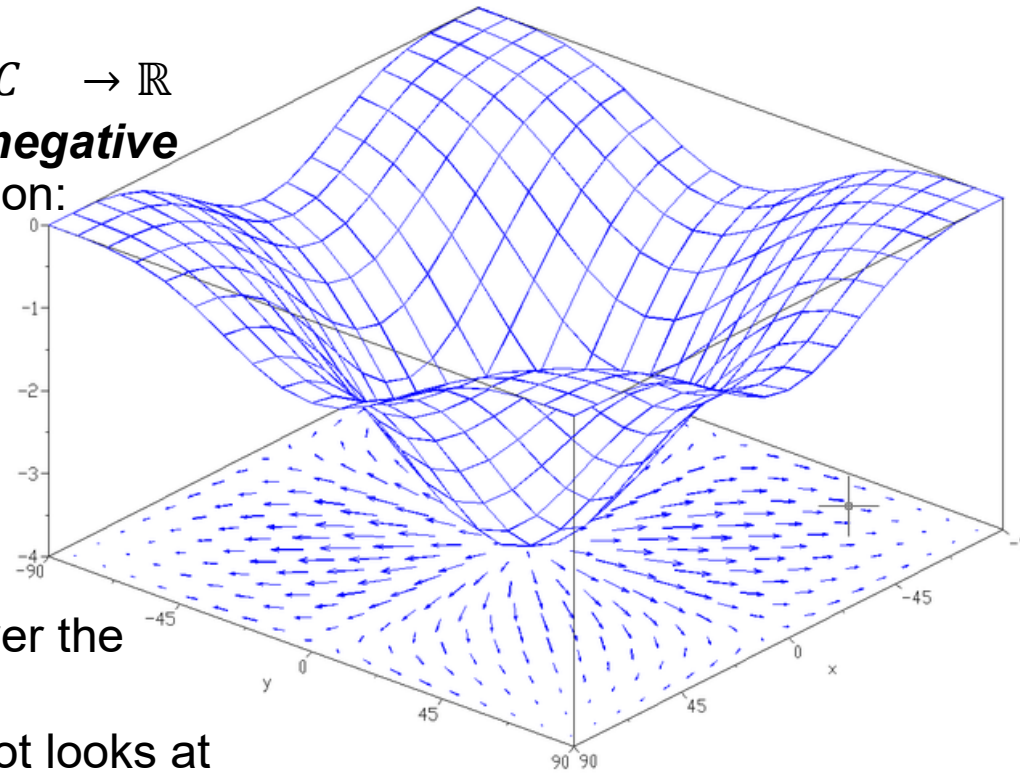
The General Idea

- Both the bowl and the spring analogies are ways of storing *potential energy*
- The robot aims to move to a lower energy configuration
- A ***potential function*** is a function $U\colon C \quad \to \mathbb{R}$
- Energy is minimized by following the ***negative gradient*** of the potential energy function:

$$\nabla U(\boldsymbol{q}) = \begin{bmatrix} \dfrac{\partial U}{\partial q_1} \\ \vdots \\ \dfrac{\partial U}{\partial q_M} \end{bmatrix}$$



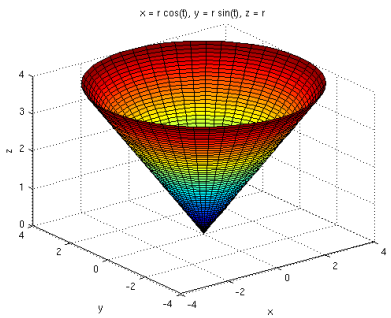
- We can now think of a ***vector field*** over the configuration space
  - At any given point in time, the robot looks at the vector corresponding to its current configuration and moves in that direction

Attractive/Repulsive Potential Field

$$U(\boldsymbol{q}) = U_{att}(\boldsymbol{q}) + U_{rep}(\boldsymbol{q})$$

- U_{att} is the "attractive" potential --- move to the goal
- U_{rep} is the "repulsive" potential --- avoid obstacles

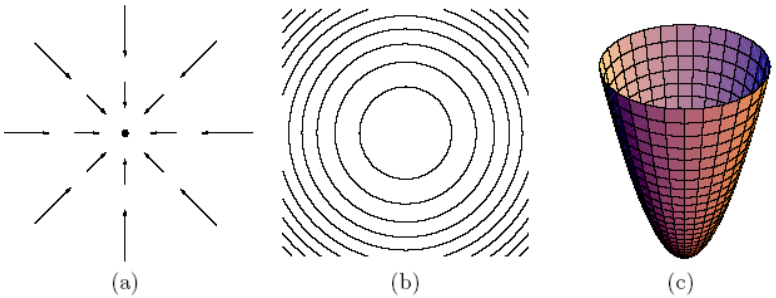
Artificial Potential Field Methods: Attractive Potential



Conical Potential

$$U(q) = \zeta d(q, q_{\text{goal}}).$$

$$\nabla U(q) = \frac{\zeta}{d(q, q_{\text{goal}})}(q - q_{\text{goal}}).$$



Principles of Robot Motion

Quadratic Potential

$$U_{\text{att}}(q) = \frac{1}{2}\zeta d^2(q, q_{\text{goal}}),$$

$$\begin{aligned}\nabla U_{\text{att}}(q) &= \nabla\left(\frac{1}{2}\zeta d^2\left(q, q_{\text{goal}}\right)\right), \\ &= \frac{1}{2}\zeta \nabla d^2(q, q_{\text{goal}}), \\ &= \zeta(q - q_{\text{goal}}),\end{aligned}$$

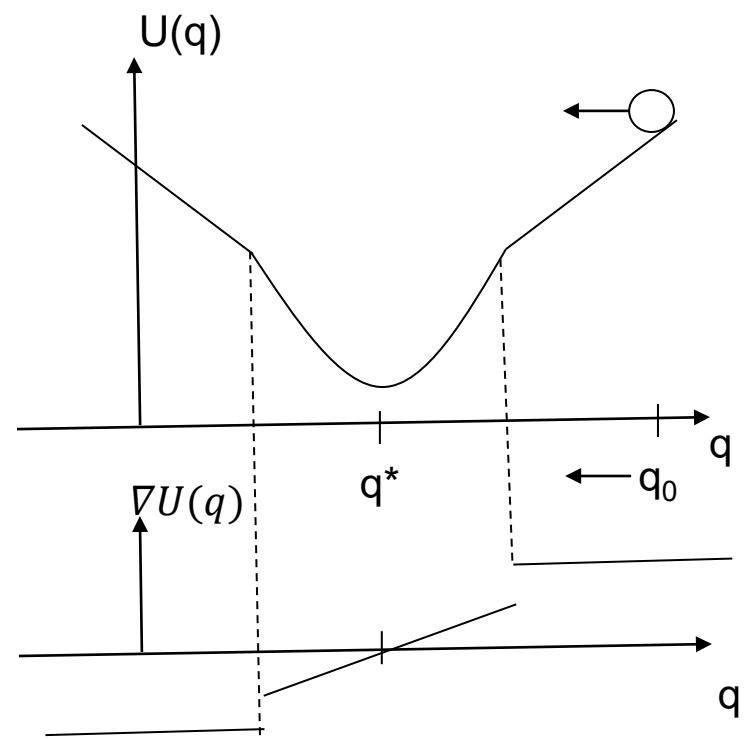
ζ is a parameter used to scale the attractive potential

Artificial Potential Field Methods: Attractive Potential

Combined conic and quadratic potential functions

$$U_{\text{att}}(q) = \begin{cases} \frac{1}{2}\zeta d^2(q, q_{\text{goal}}), & d(q, q_{\text{goal}}) \leq d^*_{\text{goal}}, \\ d^*_{\text{goal}}\zeta d(q, q_{\text{goal}}) - \frac{1}{2}\zeta (d^*_{\text{goal}})^2, & d(q, q_{\text{goal}}) > d^*_{\text{goal}}. \end{cases}$$

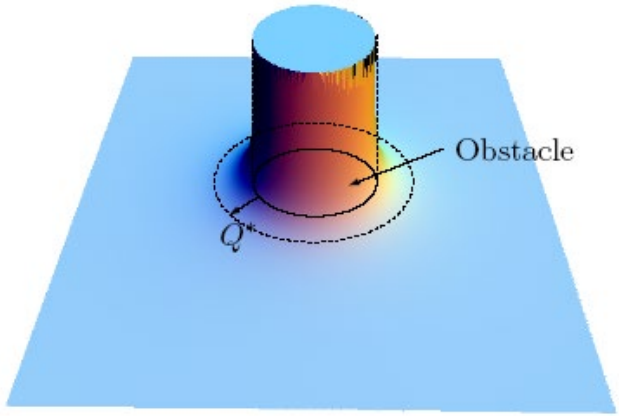
$$\nabla U_{\text{att}}(q) = \begin{cases} \zeta(q - q_{\text{goal}}), & d(q, q_{\text{goal}}) \leq d^*_{\text{goal}}, \\ \frac{d^*_{\text{goal}}\zeta(q - q_{\text{goal}})}{d(q, q_{\text{goal}})}, & d(q, q_{\text{goal}}) > d^*_{\text{goal}}, \end{cases}$$



In some cases, it may be desirable to have distance functions that grow more slowly to avoid huge velocities when far from the goal

one idea is to use the quadratic potential near the goal (< d*) and the conic farther away
One minor issue: what?

The Repulsive Potential



Principles of Robot Motion

$D(\boldsymbol{q})$: distance to the closest obstacle
Q^*: maximum repulsive distance
η: repulsive field gain

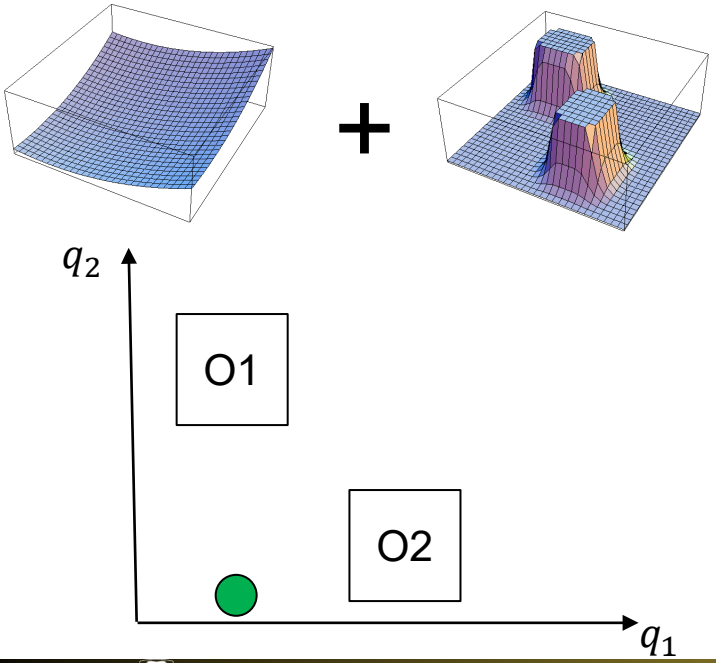
$$U_{\text{rep}}(q) = \begin{cases} \frac{1}{2}\eta\left(\frac{1}{D(q)} - \frac{1}{Q^*}\right)^2, & D(q) \leq Q^*, \\ 0, & D(q) > Q^*, \end{cases}$$

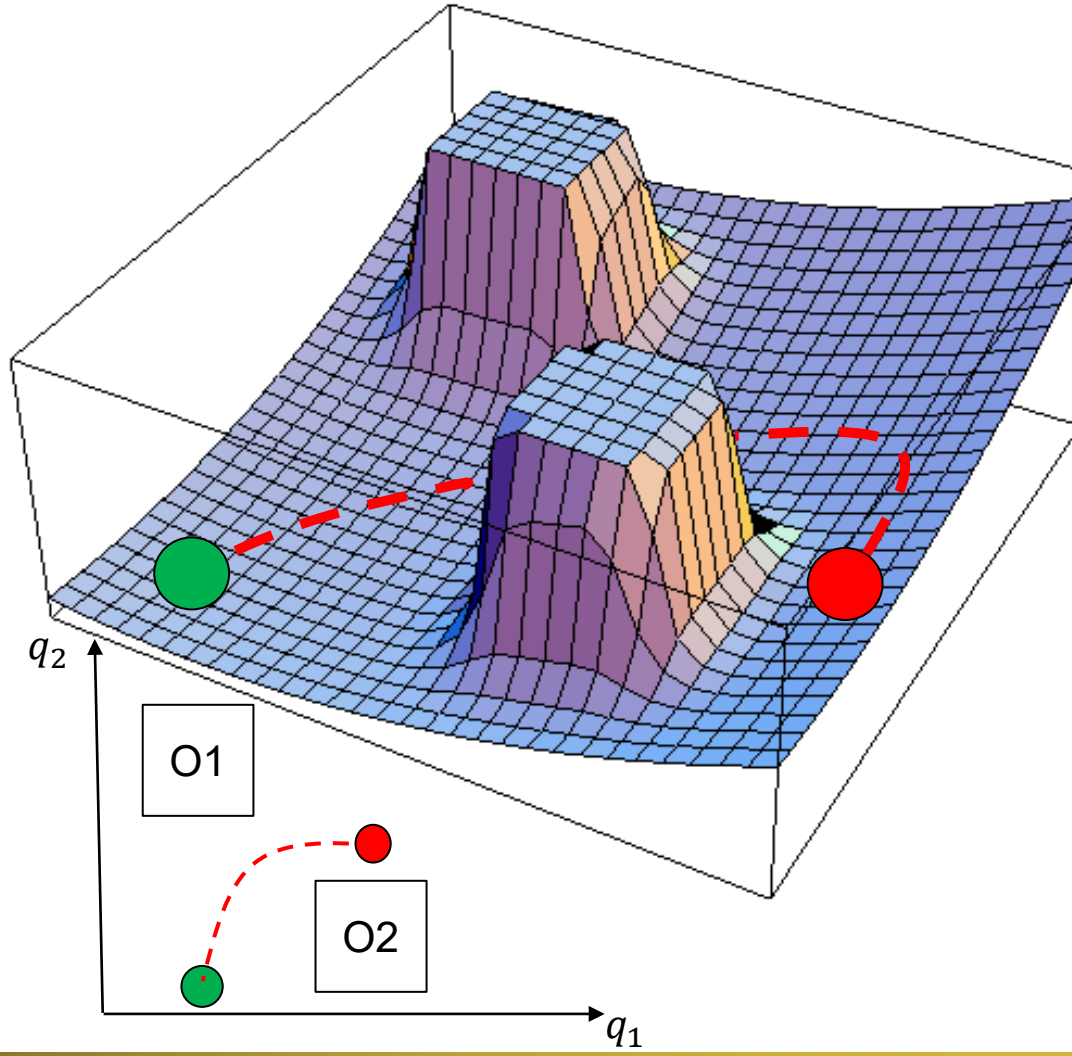
whose gradient is

$$\nabla U_{\text{rep}}(q) = \begin{cases} \eta\left(\frac{1}{Q^*} - \frac{1}{D(q)}\right)\frac{1}{D^2(q)}\nabla D(q), & D(q) \leq Q^*, \\ 0, & D(q) > Q^*, \end{cases}$$

Total Potential Function

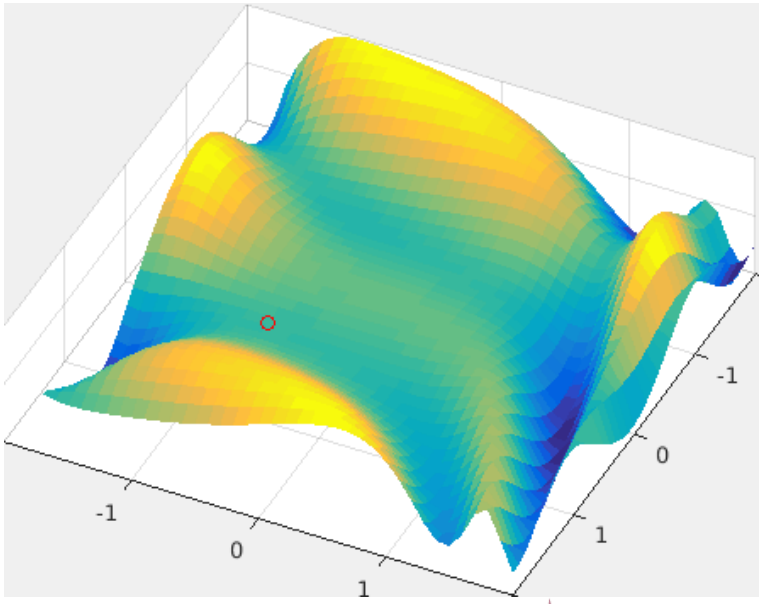
$$U(q) = U_{\text{att}}(q) + U_{\text{rep}}(q)$$

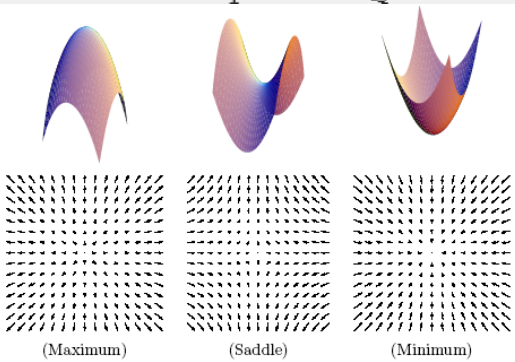




Gradient Descent

- A simple way to get to the bottom of a potential function $U(\boldsymbol{q})$is to follow the gradient $-\nabla U(\boldsymbol{q})$
- At each configuration $\boldsymbol{q}$, evaluate $-\nabla U(\boldsymbol{q})$ and take a "small step" $\Delta\boldsymbol{q}$ in that direction to reach a new configuration $\boldsymbol{q} + \Delta\boldsymbol{q}$
- At a critical configuration $\boldsymbol{q}^*$, $\nabla U(\boldsymbol{q}^*) = 0$





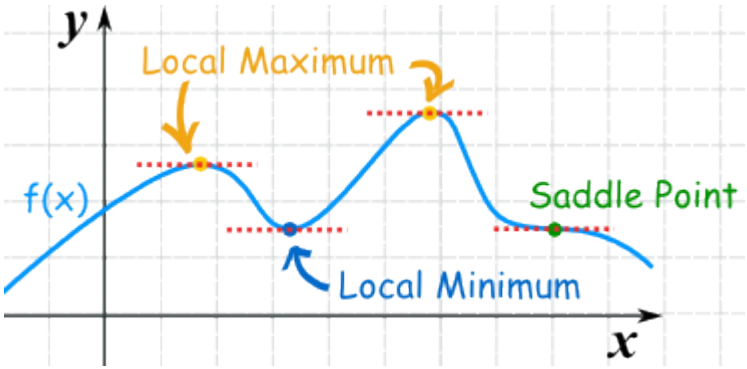
Principles of Robot Motion

Gradient Descent

- For a 1-d function, how do we know we are at a minimum (or maximum)?
  - 1st derivative tells us if we have a critical point (maximum, minimum, saddle point)
  - 2nd derivative tells us the kind of critical point

  If $f''(x^*) < 0$ then x^* is a maximum

  If $f''(x^*) > 0$ then x^* is a minimum
- For a M-d function the Hessian $H_f(\boldsymbol{x})$ is the $M \times M$ matrix of second derivatives
- If the Hessian is nonsingular $\det(H) \neq 0$, the critical point is a unique point
  - if $H_f(\boldsymbol{x}^*)$ is positive definite $(\boldsymbol{x}^{*T} H_f(\boldsymbol{x}^*)\boldsymbol{x}^* > 0)$, is a minimum (positive eigenvalues)
  - if $H_f(\boldsymbol{x}^*)$ is negative definite $(\boldsymbol{x}^{*T} H_f(\boldsymbol{x}^*)\boldsymbol{x}^* < 0)$, is a maximum (negative eigenvalues)
  - if $H_f(\boldsymbol{x}^*)$ is indefinite, $\boldsymbol{x}^*$ is a saddle point



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Wikipedia.org

The Hessian

- The Hessian is the $M \times M$ matrix of second derivatives

$$H_U(\mathbf{q}) = \begin{bmatrix} \dfrac{\partial^2 U}{\partial q_1^2} & \cdots & \dfrac{\partial^2 U}{\partial q_1 q_M} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 U}{\partial q_1 q_M} & \cdots & \dfrac{\partial^2 U}{\partial q_M^2} \end{bmatrix}$$

- The Hessian is the Jacobian of the gradient

$$H_U(\boldsymbol{q}) = J_{\nabla U}(\boldsymbol{q})$$

Gradient Descent

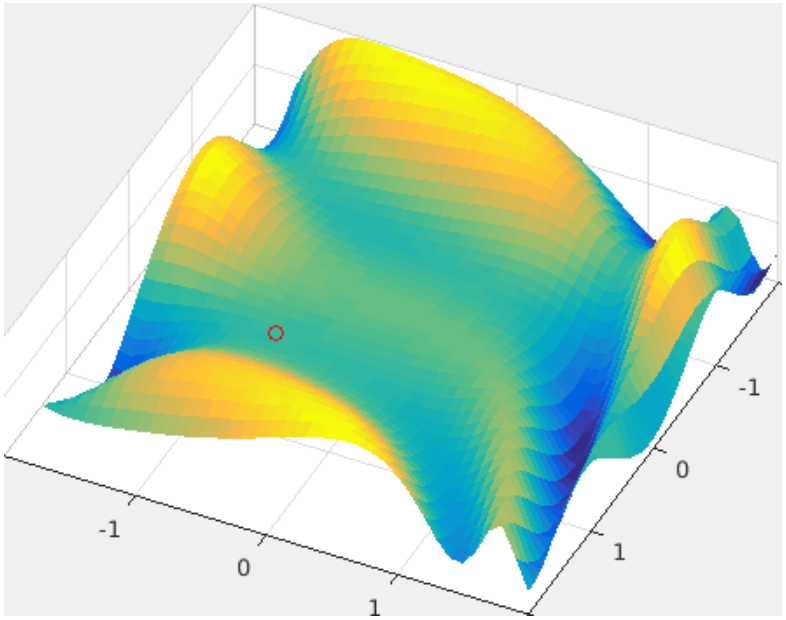
Algorithm

1. You need:

   A start configuration: q_{start}

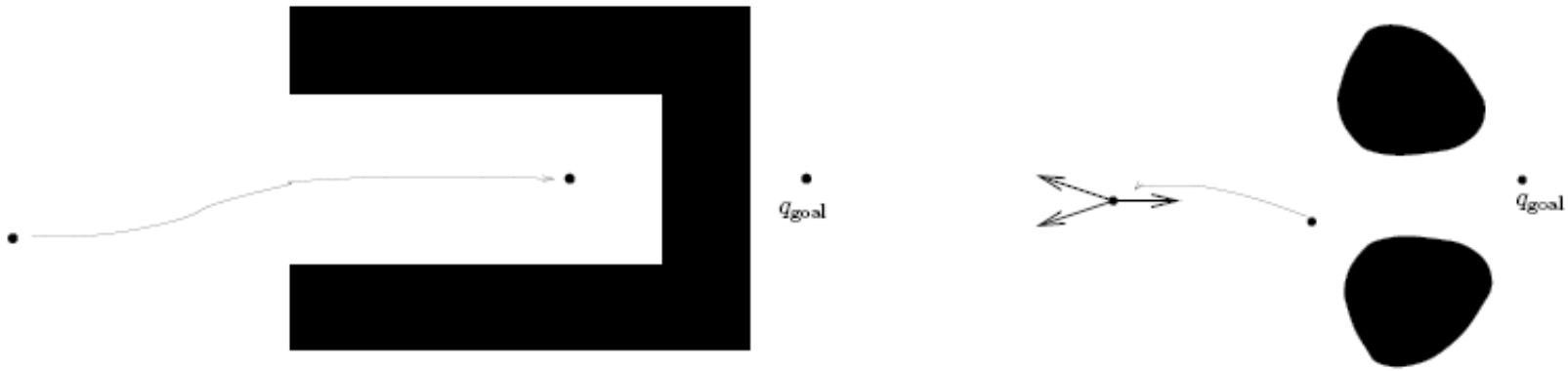
   Gradient of a potential function: $\nabla U(q)$

2. $q_0 = q_{start}$
3. $i = 0$
4. while $\nabla U(q_i) \neq 0$ do
5. $\quad q_{i+1} = q_i - \alpha_i \nabla U(q_i)$
6. $\quad i = i + 1$



Potential Functions Question

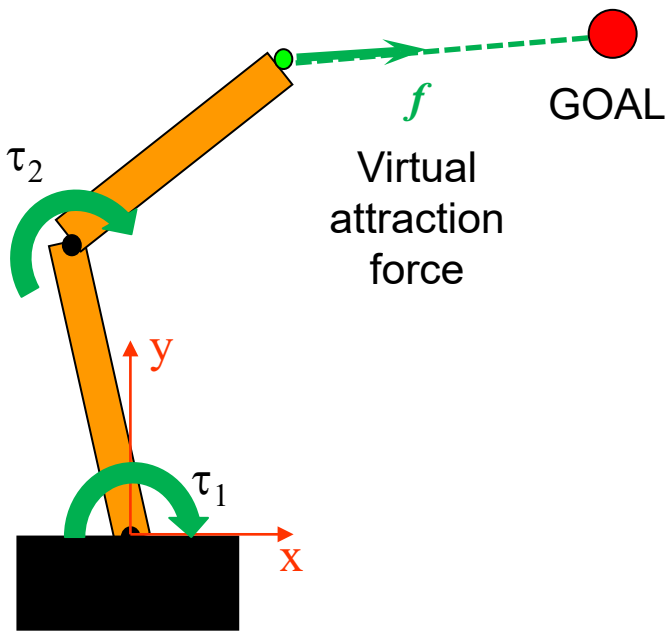
- How do we know that we have only a single (global)



Principles of Robot Motion

- We have two choices:
  - not guaranteed to be a global minimum: do something other than gradient descent (what?)
  - make sure only one global minimum

Potential Fields on Non-Euclidean Spaces



Potential Fields on Non-Euclidean Spaces

- Define a generalized force by a linear component (pure force) and an angular component (pure moment).

$$F = \begin{bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{\tau} \end{bmatrix}$$

$\boldsymbol{f} \in \mathbb{R}^3$ linear component

$\boldsymbol{\tau} \in \mathbb{R}^3$ rotational component

- The net work by a force applied at frame Hand is

$$W = \int_{t_1}^{t_2} {}^{B}V_H^b \cdot F_H dt$$

- Work is preserved and must be equal by an equivalent force applied at frame Base.

$$F_H = \mathrm{Ad}^T F_B$$

Potential Fields on Non-Euclidean Spaces

- Let ${}^{B}E_H(\boldsymbol{q}(t))$ be the time varying forward kinematics.
- The net work by applying a generalized force F_H at the Hand is

$$W = \int_{t_1}^{t_2} {}^{B}V_H^b \cdot F_H dt$$

- Work is the same as the one performed by the joints

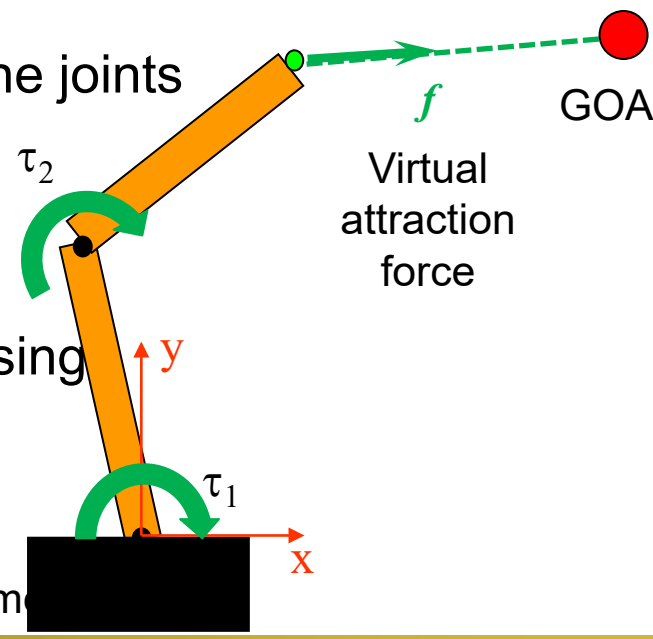
$$W = \int_{t_1}^{t_2} \dot{\boldsymbol{q}}(t) \cdot \boldsymbol{\tau} dt = \int_{t_1}^{t_2} {}^{B}V_H^b \cdot F_H dt$$

- This means that $\dot{\boldsymbol{q}}(t) \cdot \boldsymbol{\tau} = {}^{B}V_H^b \cdot F_H$ and by using the manipulator Jacobian we get

$$\boldsymbol{\tau} = J(\boldsymbol{q})^T F_S$$

joint torque/force

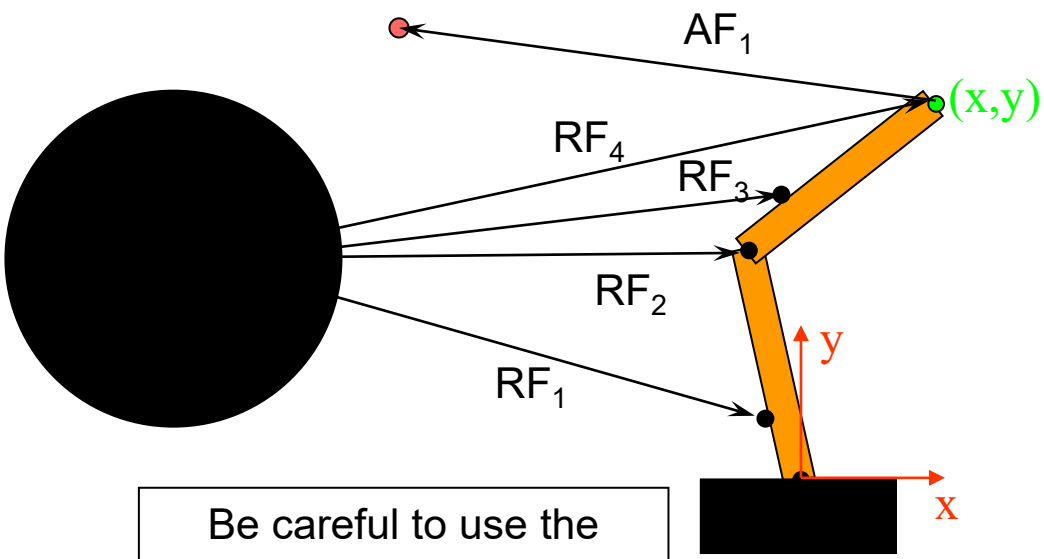
Cartesian force/mom



Add Obstacles

- Pick several points on the manipulator
- Compute attractive and repulsive virtual forces for each
- Transform these forces into the configuration space and add them
- Use the resulting torques to move the robot (in its configuration space)

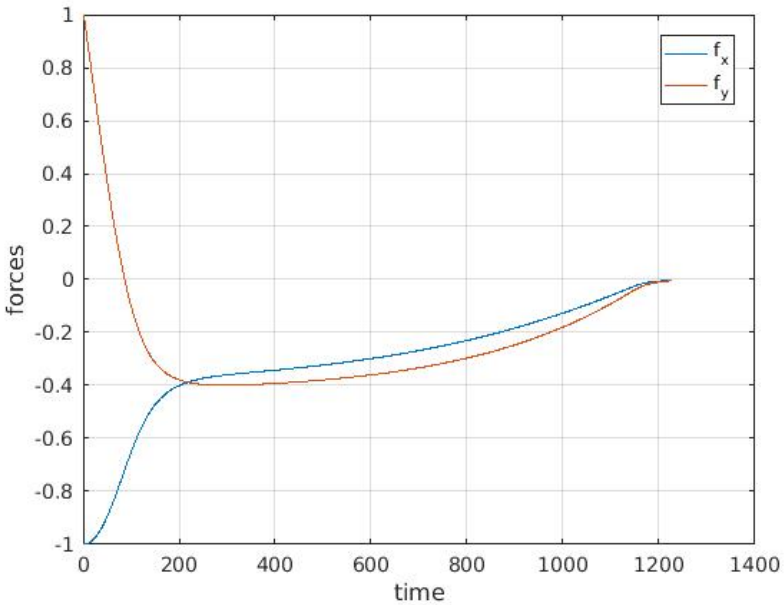
AF: attraction force
RF_i: repulsion force



Be careful to use the correct Jacobian!
Thus far we have been using the Jacobian that concern the coordinate frame of the end-effector

2D Stick Robot

Forces representing the gradient of a potential function in Cartesian space



Initial force is $[-1\ 1]^T$ indicating that the end-effectors needs to move along –X and +Y with equal magnitude

Joint torques that will generate the desired forces