3 Important Routing Parameters

In order for a graph $G = (V, E)$ to have a good routing performance, the following two properties should be fulfilled:

- Its diameter should be small, since it is a lower bound for the worst case time necessary to send a message from one node to another.
- The graph should have no "bottleneck", i.e., a small region many packets are forced to traverse. The complete binary tree, for instance, is a very bad topology for a communication network, since one half of the nodes can only communicate with the other half via the root.

In the following we will present parameters that measure these properties and describe their relationships.

3.1 The diameter

One important goal in choosing a topology for a network is that it has a small diameter. The following claim presents a lower bound for this.

Theorem 3.1 *Every graph of maximum degree $d > 2$ and size n must have a diameter of at least $\lfloor \log_{d-1} n \rfloor - 1$.*

Proof. Suppose we have a graph $G = (V, E)$ of maximum degree d and size n. Start from any node $v \in V$. In a first step at most d other nodes can be reached. In two steps at most $d \cdot (d-1)$ additional nodes can be reached. Thus, in general, in at most k steps at most

$$1 + \sum_{i=0}^{k-1} d \cdot (d-1)^i = 1 + d \cdot \frac{(d-1)^k - 1}{(d-1) - 1} \leq \frac{d \cdot (d-1)^k}{d-2}$$

nodes (including v) can be reached. This has to be at least n to ensure that v can reach all other nodes in V within k steps. Hence,

$$(d-1)^k \geq \frac{(d-2) \cdot n}{d} \quad \Leftrightarrow \quad k \geq \log_{d-1}((d-2) \cdot n/d) \ .$$

Since $\log_{d-1}((d-2)/d) > -2$ for all $d > 2$, this is true only if $k \geq \lfloor \log_{d-1} n \rfloor - 1$. □

Theorem 3.1 uses as a construction for the lower bound a complete $(d-1)$-ary tree with a root of degree d. However, it is easy to see that in this tree there are two nodes (see the leaves v and w in Figure 1) with a distance of approximately $2 \log_{d-1} n$, which is by a factor of 2 larger than the lower bound. Can networks with a better diameter be constructed? The next theorem gives an answer to this.

Theorem 3.2 *For every even $d > 2$ there is an infinite family of graphs G_n of maximum degree d and size n with a diameter of at most $\log_{d/2} n$.*

Proof. The proof is part of the assignment. □

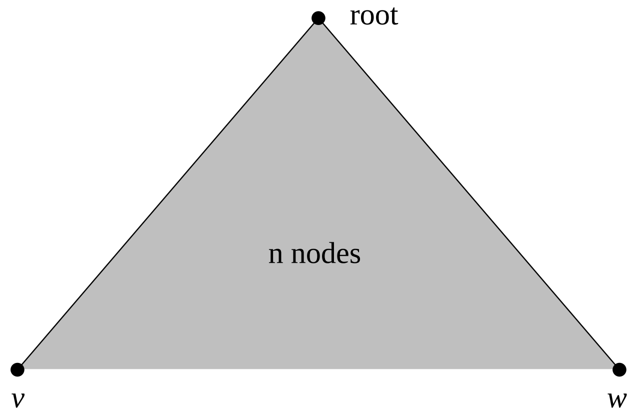

Figure 1: Nodes with highest distance in a tree.

3.2 The expansion

To measure the connectivity of a graph, the following parameters are usually used. For every $U \subseteq V$ let $\bar{U} = V \setminus U$, and for every pair of sets $U, U' \subseteq V$ let $C(U, U')$ be the set of all edges in E that have one endpoint in U and the other endpoint in U'. The *bisection width* of G is defined as

$$\min_{u \subseteq V, |U| = \lfloor |V|/2 \rfloor} \frac{|C(U, \bar{U})|}{|U|} .$$

A more general form is the *expansion* of a graph G. It is often defined as

$$\min_{u \subseteq V, |U| \leq \lfloor |V|/2 \rfloor} \frac{|C(U, \bar{U})|}{|U|} .$$

Obviously, a graph of maximum degree d can have an expansion of at most d. Graphs with a high expansion are called *expanders*. More formally, an (α, β, d, n)*-expander* is a d-regular graph on n nodes with the property that every subset of size at most αn has an expansion of at least β. The next theorem demonstrates that there are graphs that asymptotically reach a best possible expansion.

Theorem 3.3 *For every $d > 2$, there is an infinite class of graphs G_n that are $(1/d, d/2, d, n)$-expanders.*

Proof. It can be shown that a random d-regular graph has this property with high probability.
□

Next we give a relationship between the expansion and the diameter.

Theorem 3.4 *Every graph of size n with expansion α has a diameter of at most $2 \log_{1+\alpha} n$.*

Proof. Consider an arbitrary graph $G = (V, E)$ of size n with expansion α. In this case, we will show by complete induction that the number of nodes that can be reached in at most k steps from any node v in G is at least $(1 + \alpha)^k$.

As the base case we choose $k = 0$. For this k our assumption is obviously true. Now, assume that it is known to be true that for some k at least $(1+\alpha)^k$ nodes can be reached from v in at most k steps. Then the number of new nodes that can be reached in $k+1$ steps is at least $\alpha \cdot (1+\alpha)^k$, and therefore the total number of nodes that can be reached in at most $k+1$ steps is at least

$$(1+\alpha)^k + \alpha \cdot (1+\alpha)^k = (1+\alpha)^{k+1} ,$$

which proves our assumption.

Now, consider any two nodes $v, w \in V$ that have a maximum distance in G. For $k > \log_{1+\alpha}(n/2)$, the number of nodes that can be reached in at most k steps from v and w is at least $(1+\alpha)^k > n/2$. Thus, there must be a node that can be reached both from v and from w in at most k steps, which proves the claim. □

3.3 The routing number

A parameter that somehow combines both the diameter and the expansion is the so-called routing number (see [Scheideler]). Consider an arbitrary graph $G = (V, E)$ of size n, i.e., $|V| = n$. Let S_n be the set of all permutations on $[n]$. For a permutation $\pi \in S_n$, let $R(G, \pi)$ be the minimum possible number of steps required to route packets in G according to π. Then the *routing number* $R(G)$ of G is defined as

$$R(G) = \max_{\pi \in S_n} R(G, \pi) .$$

In the case that there is no risk of confusion about the graph G, we will simply write R instead of $R(G)$. The definition of the routing number and Fact 2.1 immediately imply the following result.

Fact 3.5 *For any graph G of size n and routing number R, there exists a collection of paths for any permutation routing problem with congestion and dilation at most R.*

Thus, every graph of routing number R must have a diameter of at most R. Furthermore, the following result is known.

Theorem 3.6 (Leighton, Rao) *Any graph G of size n with expansion α has a routing number R with $\Theta(\alpha^{-1}) \leq R \leq \Theta(\alpha^{-1} \log n)$.*

Proof. We will only show that $R \geq \Theta(\alpha^{-1})$. Due to the definition of α, there must exist a set $U \subseteq V$ with the properties that $|U| \leq |V|/2$ and $|C(U, \bar{U})| = \alpha \cdot |U|$. We choose now a permutation for which all requests starting in a node in U have their destination outside of U (this is possible, since $|U| \leq |V|/2$). In this case, $|U|$ packets have to traverse the cut $C(U, \bar{U})$. Since this cut only consists of $\alpha|U|$ edges, the time needed for this is at least $|U|/(\alpha|U|) = \alpha^{-1}$. Hence, $R \geq \alpha^{-1}$. □

3.4 References

- T. Leighton and S. Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *Journal of the ACM* 46(6):787–832, 1999.
- C. Scheideler. Universal Routing Strategies for Interconnection Networks. Lecture Notes in Computer Science 1390, Springer, 1998.