Complexity Theory
Spring 2003
Midterm Exam

Problem 1: Complexity Classes (8 points)
1. Show that BPP ⊆ PSPACE.
2. Show that NPSPACE ⊆ EXP. (Hint: look at Lemma 3.6 and Savitch’s Theorem.)
3. Consider any $C$-complete language $L$ for some complexity class $C$. Show that for every language $L' \in C$ with $L \leq_p L'$ it holds that also $L'$ is $C$-complete.
4. Show that the language SDE (solvable Diophantine equations) restricted to variables $x$ in the range $\{0, \ldots, K\}$ for some fixed $K$ is in NP.

Problem 2: Complexity of Logic (6 points)
1. Show that for all Boolean expressions $\phi_1, \phi_2,$ and $\phi_3$ it holds that
   \[ ((\phi_1 \lor \phi_2) \land \phi_3) \equiv ((\phi_1 \land \phi_3) \lor (\phi_2 \land \phi_3)) \, . \]
2. Show that for all quantified Boolean expressions $\phi_1$ and $\phi_2$ it holds that
   \[ \exists x (\phi_1 \lor \phi_2) \equiv (\exists x \phi_1 \lor \exists x \phi_2) \, . \]
3. Transform $\forall x_1 \forall x_2 \exists x_3 \forall x_4 \left( (x_1 \land (x_2 \lor \neg x_4)) \land (x_3 \lor x_4) \right)$ into an equivalent arithmetical sentence with variables $x \in \{0, 1\}$. (A proof of equivalence is not needed.)

Problem 3: Undecidability (2 points)
Use the fact that the language
\[ L = \{ \langle M \rangle w \mid M \text{ started with } w \text{ accepts} \} \]
is undecidable to show that also the language
\[ E = \{ \langle M \rangle \mid M \text{ is a TM and } |L(M)| = 1 \} \]
is undecidable.

Problem 4: Complexity of Arithmetics (4 points)
Give elementary arithmetical expressions for the following statements:
1. $x \in \mathbb{N}$ encodes a word $w \in \Sigma^*$ (in $p$-ary form) that contains the symbol $c$.
2. $x \in \mathbb{N}$ is a prime number.

Good luck!