Problem 1: Diverse Topics  (8 points)
(a) Show that $|\mathbb{N}| = |\mathbb{N}^3|$. (2 points)
(b) Show that if $L_1$ and $L_2$ are recursively enumerable then also $L_1 \cap L_2$ is recursively enumerable. (2 points)
(c) Show that there is an infinite hierarchy for non-deterministic time. (2 points)
(d) Give an elementary arithmetical expression for the statement “$x$ is the smallest common multiple of $a$ and $b$”. (2 points)

Problem 2: Complexity Classes  (4 points)
Recall that $C^O$ is the class of all decision problems that are in $C$ if the Turing machines are allowed to use oracle $O$. Our aim is to show that $P^{QSAT} = NP^{QSAT}$. We do this in two steps.
(a) Show that $P^{QSAT} = PSPACE$. (Hint: remember to show “$\subseteq$” in both directions, and use the fact that QSAT is PSPACE-complete.) (2 points)
(b) Show that $NP^{QSAT} = PSPACE$. (Hint: extend the proof of $NP \subseteq PSPACE$.) (2 points)

Problem 3: Undecidability  (3 points)
Use the fact that the language $L_a = \{\langle M \rangle w \mid M$ started with $w$ accepts$\}$ is undecidable to show that also the language
\[ E = \{\langle M_1 \rangle \langle M_2 \rangle \mid L(M_1) = L(M_2)\} \]
is undecidable. (Make sure your proof is complete. That is, just sketching $f$ is not sufficient.)

Problem 4: Logic  (5 points)
(a) Show that for all quantified Boolean expressions $\phi_1$ and $\phi_2$ it holds that (2 points)
\[ \forall x (\phi_1 \land \phi_2) \equiv (\forall x \phi_1 \land \forall x \phi_2) . \]
(b) Transform the expression
\[ (\exists x_1((x_1 \land x_3) \lor \forall x_2(x_1 \land x_2))) \land (\forall x_3(x_3 \lor \exists x_4(x_1 \lor x_4))) \]
into prenex normal form using the equivalences in Section 4. (Hint: Be careful about the variables a quantifier applies to when moving it outside of the Boolean expression. Sometimes, substitutions have to be used to maintain equivalence.) (3 points)

Good luck!