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Complexity Theory

Spring 2004

Final Exam

Problem 1: Interactive Proofs (6 points)

(a) Suppose that in an interactive proof the verifier is not allowed to use random bits. Show that in this case, IP($poly$) $\subseteq$ NP. (2 points)

(b) Consider PCPs with one of the following restrictions:

- (i) For every input, the verifier only asks at most once for any bit in the proof string over all of its possible computations.
- (ii) For every input, the verifier only asks for polynomially many different bits of the proof string over all of its possible computations.

Show that in both cases it holds that PCP($poly, poly$) $\subseteq$ PSPACE. (4 points)

Problem 2: Approximability (8 points)

(a) Suppose that for two minimization problems A and B, there is an L-reduction from A to B with $\alpha = 2$ and $\beta = 3$ and there is a polynomial time 2-approximation algorithm for B. How well can A be approximated in polynomial time? (2 points)

(b) In the 3COL problem we are given a graph $G = (V, E)$ and the problem is to assign a color out of three possible colors to each node so that the number of nodes v with a different color than all of their neighbors (i.e. all w with $\{v, w\} \in E$) is maximized. Let this maximum quantity be defined as $\gamma(G)$. In 3COL(ϵ) we are given the promise that either $\gamma(G) = n$ or $\gamma(G) < n/(1 + \epsilon)$ where $n = |V|$. Let us only consider graphs of degree at most 4. Show that in this case, 3COL(ϵ) $\in$ PCP($log, O(1)$) for any constant ϵ. (I.e. construct a PCP for 3COL and justify why it fulfills the PCP requirements.) (4 points)

(c) It is known that there is an L-reduction from MAX3SAT to 3COL. What consequences does this have for the approximability of 3COL? Justify your answer. (2 points)

Problem 3: Quantum Computing (9 points)
Consider the apparatus given in Figure 1. Let

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$$

and consider an observable O with four subspaces E_1, E_2, E_3, and E_4. We start with $|X\rangle = |00\rangle$.

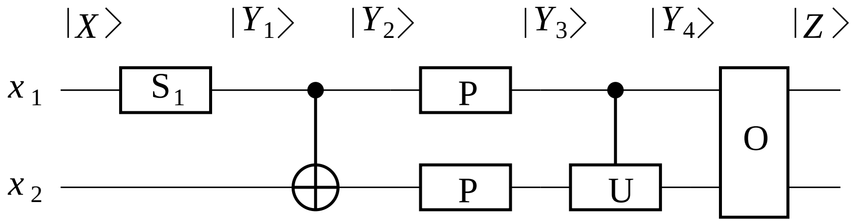

Figure 1: A 2-qubit Quantum circuit.

(a) Compute $|Y_1\rangle$. (1 point)

(b) Compute $|Y_2\rangle$. (1 point)

(c) Compute $|Y_3\rangle$. (2 points)

(d) Compute $|Y_4\rangle$. (1 point)

(e) Let $B_1 = \{|00\rangle\}$ be the basis of E_1, $B_2 = \{|01\rangle\}$ be the basis of E_2, $B_3 = \{\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)\}$ be the basis of E_3, and $B_4 = \{\frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)\}$ be the basis of E_4. Compute all possible outcomes for $|Z\rangle$ and compute the probabilities with which these outcomes occur. (4 points)

Problem 4: Dynamical Systems (4 points)
Consider a linear dynamical system with the following transition graph:

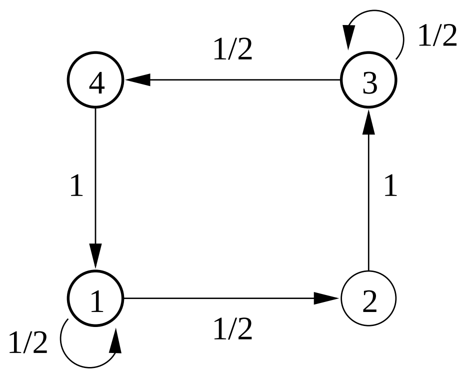

Figure 2: A 4-state transition graph.

(a) Show that the corresponding transition matrix P is ergodic (i.e. there is a $t > 0$ so that $P^t > 0$). (2 points)

(b) Find the unique stationary stochastic distribution for P. (2 points)

Good luck!