

## Modern Complexity Theory

Spring 2005

### Assignment 9

Any *special* unitary  $2 \times 2$  matrix (i.e., with unit determinant) can be expressed as

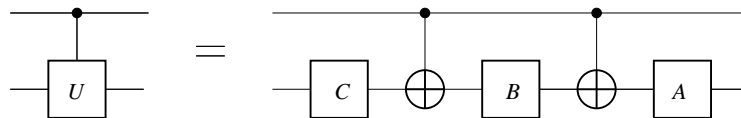
$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}.$$

Consider the following set of standard matrices:

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad R_z(\alpha) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

**Problem 27** (5 points):

Show that for any special unitary matrix  $U$ , the matrices  $A = R_z(\alpha) \cdot R_y(\theta/2)$ ,  $B = R_y(-\theta/2) \cdot R_z(-(\alpha+\beta)/2)$ , and  $C = R_z((\beta-\alpha)/2)$  have the property that  $A \cdot B \cdot C = I$  and  $A \cdot \sigma_x \cdot B \cdot \sigma_x \cdot C = U$ . Use this to show that for any special unitary matrix  $U$  there exist unitary matrices  $A$ ,  $B$ , and  $C$  such that the controlled  $U$ -gate can be simulated by a network of the form



(The vertical line with the  $\oplus$  represents a controlled not gate. Recall that  $e^{i\delta} = \cos \delta + i \sin \delta$ .)

**Problem 28** (5 points):

Show that for any unitary  $2 \times 2$  matrix the  $U$ -gate controlled by the logical AND of two qubits can be simulated by a network of the following form with  $V^2 = U$ .

