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Complexity Theory

Spring 2001

Midterm Exam

Problem 1: Oracle Computations (7 points)

1. Show that for any infinite class of recursively enumerable languages $\mathcal{C} = \{L_1, L_2, L_3, \ldots\}$ and any oracle O, $|\mathcal{C}| = |\mathcal{C}^O|$. Hints:
   - First, show that $|\{0,1\}^*| \geq |\mathcal{C}^O|$ (1 point)
   - and that $|\mathcal{C}| \geq |\{0,1\}^*|$, (1 point)
   - and then use these to show that $|\mathcal{C}| = |\mathcal{C}^O|$. (1 point)
2. For any class of languages $\mathcal{C}$ let co-$\mathcal{C} = \{\bar{L} :\ L \in \mathcal{C}\}$. Show that NP $\cup$ co-NP $\subseteq P^{SAT}$. (2 points)
3. Show (with the help of the fact PSPACE = NPSPACE) that $\mathrm{P}^{QSAT} = \mathrm{NP}^{QSAT}$. (2 points)

Problem 2: Undecidability (3 points)
A set of recursively enumerable languages is called *trivial* if it is either empty or consists of all recursively enumerable languages. Show that for any nontrivial $\mathcal{L}$ the language

$$L_{\mathcal{L}} = \{\langle M \rangle :\ L(M) \in \mathcal{L}\}$$

is not decidable.
Hints: Since $\mathcal{L}$ is non-trivial, there must be a r.e. language L_1 that belongs to $\mathcal{L}$ and a r.e. language L_2 that does not belong to $\mathcal{L}$. Without loss of generality, we may choose $L_2 = \emptyset$ (otherwise consider $\bar{\mathcal{L}}$). Use this fact together with the language $L_u = \{\langle M \rangle w : M \text{ accepts } w\}$ to construct a suitable reduction.

Problem 3: Complexity Hierarchies (3 points)
Show that for any $k \in \mathbb{N}$ and $\epsilon > 0$,

$$\mathrm{NSPACE}(n^k) \subset \mathrm{NSPACE}(n^{k+\epsilon}) .$$

Hint: use Savitch's theorem and as a fact that for any space-constructible functions $s_1(n), s_2(n)$ and $f(n) \geq n$ it holds: if

$$\mathrm{NSPACE}(s_1(n)) \subseteq \mathrm{NSPACE}(s_2(n))$$

then

$$\mathrm{NSPACE}(s_1(f(n))) \subseteq \mathrm{NSPACE}(s_2(f(n))) .$$

Problem 4: NP completeness (7 points)

1. Give the definition of an NP-complete language. (1 point)

2. Show that the language

$$3SAT = \{\phi : \ \phi \text{ is a satifiable Boolean expression in 3CNF}\}$$

   is NP-complete (3CNF: all clauses have exactly 3 literals).

   Hints:

   - Prove that (2 points)

$$x_1 \vee x_2 \equiv (x_1 \vee x_2 \vee y) \wedge (x_1 \vee x_2 \vee \bar{y})$$

   - and that for any $k \geq 4$, (2 points)

$$x_1 \vee x_2 \vee \ldots \vee x_k \equiv \exists y_2 \ldots \exists y_{k-2} \ (x_1 \vee x_2 \vee y_2) \wedge \left(\bigwedge_{i=2}^{k-3} (\bar{y}_i \vee x_{i+1} \vee y_{i+1}) \right) \wedge (\bar{y}_{k-2} \vee x_{k-1} \vee x_k) \ ,$$

   - and use these two facts to perform a suitable reduction with the help of SAT. (2 points)

Good luck!