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# Theory of Network Communication

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## Solutions to Assignment 6

**Problem 13** (5 points)**:**
The maximum number of times a block is replaced from 1 to $n$ disks is at most $\lfloor \log n \rfloor + 2$.

**Proof.** For any $i \in \{1, \ldots, n\}$ and any data block $b$, let $h_i(b) \in [0, 1/i]$ denote the *height* of $b$ at the presence of $i$ disks. Initially, $h_1(b) = h(b)$, and $h_i(b)$ is determined by the position in $[0, 1/i]_j$ of the disk $j$ that contains the range including $h(b)$. Now, consider any block $b$ on disk $D_{m_0}$. This block has to be moved to disk $D_{m_1}$ when $1/m_1 \leq h_{m_0}(b)$ for the first time. Hence, $m_1 = \lceil 1/h_{m_0}(b) \rceil$. Furthermore, the height of block $b$ on disk $D_{m_1}$ is $h_{m_1}(b) \leq \frac{(m_1 - m_0)}{m_1(m_1-1)}$. The next time $b$ will be moved will be to a disk $D_{m_2}$ with $m_2 \geq \lceil \frac{m_1(m_1-1)}{(m_1-m_0)} \rceil$. Now, consider the function $f(x) = x(x-1)/(x-m)$. It holds that

$$f'(x) = \frac{(2x-1)(x-m) - (x^2 - x)}{(x-m)^2} = \frac{x^2 - 2xm + m}{(x-m)^2} .$$

For $x > m$ it holds that $f'(x) = 0$ if and only if $x^2 - 2xm + m = 0$. This has the solutions

$$x_{1/2} = m \pm \sqrt{m^2 - m} .$$

Only $x_2 = m + \sqrt{m^2 - m}$ works for $x > m$. Furthermore, $f''(x_2) > 0$, which means that $f(x_2)$ is a minimum. Thus, restricted to integral values, either $\lfloor x_2 \rfloor = 2m - 1$ or $\lceil x_2 \rceil = 2m$ must be a place with a minimum. It holds that

$$f(2m-1) = \frac{(2m-1)(2m-2)}{m-1} = 2(2m-1) \quad \text{and} \quad f(2m) = \frac{2m(2m-1)}{m} = 2(2m-1) .$$

Hence, for any choice of $m_0$ and $h_{m_0}(b)$, $m_2 \geq 4m_0 - 2$. Now, let $n_i$ be the $i$th position of $b$, $i \geq 0$. Then it holds that $n_2 \geq 3$ and $n_{2(i+1)} \geq 4n_{2i} - 2$. Therefore, for all $i \geq 1$,

$$n_{2i} \geq 4^{i-1} n_2 - 2 \sum_{j=0}^{i-2} 4^j \geq 3 \cdot 4^{i-1} - 6 \cdot 4^{i-2} \geq 4^{i-1} ,$$

which proves the theorem. □

For any placement scheme, the maximum number of times a block is replaced from 1 to $n$ disks must be at least $\ln n - 1$.

**Proof.** Assume that we have $m$ used blocks. To keep an even distribution of blocks from 1 to $n$ disks, $m/i$ blocks have to be moved from $i-1$ to $i$ disks for every $i \geq 2$. Hence, the total number of replacements is at least

$$\sum_{i=2}^{n} m/i \geq m(\ln n - 1) \ .$$

Thus, the average (and therefore also the worst case) number of times a block is replaced is at least $\ln n - 1$, proving the theorem. $\square$