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Theory of Network Communication

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Solutions to Assignment 4

Problem 8 (2 points)**:**
Show that the b-ary DeBruijn graph of dimension d has a degree of $2b$ and a diameter d. Express d in terms of n (the number of nodes) and b in order to show that the DeBruijn graph can be used to prove Theorem 5.8.
Proof. Let δ be the degree and D denote the diameter of the b-ary DeBruijn graph of dimension d. We start with bounds on δ and D.

1. $\delta \leq 2b$: Take any node $(x_{d-1} \dots x_0)$ in the DeBruijn graph. From the definition of the edge set it follows that $(x_{d-1} \dots x_0)$ can only be connected to nodes $(yx_{d-1} \dots x_1)$ and $(x_{d-2} \dots x_1 z)$ with $y, z \in [b]$. Hence, every node has a degree of at most $2b$.
2. $D \leq d$: Consider any two nodes $x = (x_{d-1} \dots x_0)$ and $y = (y_{d-1} \dots y_0)$. In order to get from x to y, go along the sequence of nodes $(x, z_0, z_1, \dots, z_{d-2}, y)$ with $z_i = (y_{i-1} \dots y_0 x_{d-1} \dots x_i)$. This sequence of nodes indeed forms a path in the DeBruijn graph because of the way the edges are defined. Since such a path has a length of d, the claim follows.

Notice that the size, n, of a b-ary DeBruijn graph of dimension d is equal to b^d. Hence,

$$d = \frac{\log n}{\log b}$$

and therefore

$$D \leq \frac{\log n}{\log b} \leq \frac{\log n}{\log(\delta/2)} = \frac{\log n}{\log \delta - 1} \ .$$

This fulfills the upper bound on the diameter claimed in Theorem 5.8. □

Problem 9 (2 points)**:**
Compute the edge expansion of an $n \times n$-torus. It is sufficient here to guess the right set U and to compute the value $c(U, \bar{U}) / \min\{c(U), c(\bar{U})\}$.
Proof. If n is even, then take a vertical cut that cuts the $n \times n$-torus into two equal-sized pieces U and $\bar{U}$. Both U and $\bar{U}$ contain $n^2/2$ nodes and each node has a capacity of 4. Hence, $c(U) = c(\bar{U}) = 4 \cdot n^2/2 = 2n^2$. On the other hand, $c(U, \bar{U}) = 2n$ (notice that the torus has wrap-around edges). Hence,

$$\frac{c(U, \bar{U})}{\min\{c(U), c(\bar{U})\}} = \frac{2n}{2n^2} = \frac{1}{n} \ .$$

This is also equal to the expansion of the torus if n is even.

Now, assume that n is odd. In this case, take a vertical cut that goes one step to the right at the middle to again cut the $n \times n$-torus into two equal-sized pieces U and $\bar{U}$. In this case, we get $c(U) = 4(n^2+1)/2 = 2(n^2+1)$ and $c(\bar{U}) = 4(n^2-1)/2 = 2(n^2-1)$. Furthermore, $c(U, \bar{U}) = 2(n+1)$ (we have to add one vertical wrap-around edge and n horizontal wrap-around edges). Hence,

$$\frac{c(U, \bar{U})}{\min\{c(U), c(\bar{U})\}} = \frac{2(n+1)}{2(n^2-1)} = \frac{1}{n-1} \ .$$

This is also equal to the expansion of the torus in this case. □