Dr. Christian Scheideler
Johns Hopkins University

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Theory of Network Communication

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Solutions to Assignment 3

Problem 6 (4 points)**:**
Consider the algorithm given in Figure 1 of Assignment 3. Show that for any flow problem with demands $d_i = 1$ for every i that has a feasible flow solution with paths of length at most L when using demands $d'_i = (1+\epsilon)$, it holds that the algorithm with sufficiently large queues never has to delete any flow.
Proof. Consider some fixed edge e. The original Awerbuch-Leighton (AL) algorithm maximizes $\sum_j f_j(\Delta_j(e) - f_j)$, which represents exactly the amount by which the potential at the queues at e drops when moving a normalized flow of f_j for each commodity j. When using instead the rule in Figure **??**, we have to distinguish between two cases for commodity i taken by the discrete AL-algorithm, where i is the value with maximum $\Delta_i(e)$:

- $\Delta_i(e) > 2$: In this case, we can send a flow of 1, i.e. we can fully utilize the edge e with flow from commodity i. So the potential drop is

$$(\Delta_i(e) - 1) \geq (\sum_j f_j(\Delta_j(e) - f_j)) - 1$$

  where the f_j are chosen as in the original AL-Algorithm. The last inequality holds because i maximizes $\Delta_i(e)$ and in whatever way the f_j are chosen,

$$\sum_j f_j(\Delta_j(e) - f_j) \leq \Delta_i(e) \tag{1}$$

  because the f_j must fulfill $\sum_j f_j \leq 1$.

- $\Delta_i(e)/d_i \leq 2$: In this case, we can conclude from inequality (1) that also $\sum_j f_j(\Delta_j(e) - f_j) \leq 2$. Since the discrete AL-algorithm never increases the potential at an edge, this means that its potential drop is by at most 2 worse than the drop by the original AL-algorithm.

Combining the two cases, it follows that the discrete Awerbuch-Leighton Algorithm achieves a potential drop that is at most an additive 2 worse at any edge than the potential drop achieved by the original Awerbuch-Leighton Algorithm. Taking this into account, it follows from the proof in the lecture that the total potential drop due to the movement of flow in steps 2 and 3 of the AL-algorithm is at least

$$\left(\sum_i (1+\epsilon)\bar{q}_i\right) - (1+\epsilon)^2 L \cdot K - 2 \cdot 2m$$

where m is the number of edges On the other hand, the potential increase caused by injecting new flow at step 1 of the AL-algorithm is at most

$$\sum_i \bar{q}_i \ .$$

Step 4 of the AL-algorithm can only decrease the potential. Hence, the overall potential increase in one round of the AL-algorithm is at most

$$-\epsilon \sum_i \bar{q}_i + (1+\epsilon)^2 L \cdot K + 4m \ . \tag{2}$$

This value is guaranteed to be negative (i.e. the potential decreases) if

$$\sum_i \bar{q}_i > ((1+\epsilon)^2 L \cdot K + 4m)/\epsilon \ .$$

Since flow is only sent downwards, it must hold that $\bar{q}_i$ is the maximum queue size for commodity i in any queue of the system. Because there are $2m$ queues of each commodity in the system and according to (2), Φ increases by at most $(1+\epsilon)^2 L \cdot K + 4m$ in any step, the potential is limited to

$$\Phi \leq 2m \cdot (((1+\epsilon)^2 L \cdot K + 4m)/\epsilon)^2/2 + (1+\epsilon)^2 L \cdot K + 4m \ .$$

In the worst case, all of this potential may be concentrated in a single queue. Hence, the maximum value a $\bar{q}_i(e)$ can attain is bounded by $2\sqrt{m} \cdot ((1+\epsilon)^2 L \cdot K + 4m)/\epsilon$. $\square$

Notice that the discrete AL-algorithm can also be used for arbitrary integral demands d_i, because we can simply pretend as if a commodity of demand d_i represents d_i commodities of demand 1, and then run (or better, simulate) the AL-algorithm as given in Figure 1 for this situation.