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Theory of Network Communication

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Solutions to Assignment 2

Problem 3 (1 point)**:**
Show that in the preflow algorithm, the height of every node can be at most $2n - 1$.
Proof. Consider any node $u \in V \setminus \{s\}$ and let $U \subseteq V$ be the set of all nodes reachable from u in G_f. Suppose that $s \notin U$. Then $|U| < n$, and since the graph is assumed to be connected, it must hold that

$$f(U, \bar{U}) = c(U, \bar{U}) > 0 .$$

Since

$$f(U, \bar{U}) = \sum_{v \in U} \sum_{(v,w) \in E} f(v, w) ,$$

it follows that there must be a $v \in U$ with

$$\sum_{(v,w) \in E} f(v, w) > 0 .$$

This implies that $e(v) < 0$, i.e. the excess of v must be negative, which can only be the case if $v = s$. Hence, s must be in U. Thus, for any node $u \in V$ there must be a path in G_f of length at most $n - 1$ from u to s. Using the facts that $h(s) = n$ and $h(v) \leq h(w) + 1$ for all $(v, w) \in E_f$, it follows that $h(u) \leq 2n - 1$ for all $u \in V$. □

Problem 4 (4 points)**:**
Show that for $T \geq n$ the T-balancing algorithm will converge to a fixpoint with $L_{\text{BAL}} = L_{\text{OPT}}$. To do this, revisit the notation in the proof of Theorem 3.13 and suppose that $L_{\text{BAL}} > L_{\text{OPT}}$. Take as granted that in this case there must exist two flow paths p_i and q_i with the same flow value > 0 where $\lambda_{p_i} \leq \lambda_e(P \setminus Q)$ for all edges $e \in p_i$, $\lambda_{q_i} \leq \lambda_e(Q \setminus P)$ for all edges $e \in q_i$, p_i and q_i share the same endpoints, and $\ell_{p_i} > \ell_{q_i}$.
In this case, argue that we can extract two path pieces p_i' and q_i' from p_i and q_i so that p_i' and q_i' are node-disjoint apart from their endpoints and $\ell_{p_i'} > \ell_{q_i'}$. From here, conclude that the average δ_e of an edge in q_i' must be at least $\ell_{p_i'} \cdot T \cdot d/(\ell_{p_i'} - 1)$, and therefore there must exist an edge e in q_i' with $\delta_e \geq (T + 1)d$ if $T \geq n$. However, in this case the edge cannot belong to q_i' (see the definition of q_i).
Proof. We construct p_i' and q_i' by pruning p_i and q_i until we find a suitable pair p_i' and q_i'. The pruning strategy is easy: We start with the starting point of p_i and q_i. In each pruning step, we follow p_i and q_i until they meet again for the first time in some node, say, v. If for the path pieces p_i' and q_i' till v it holds that $\ell_{p_i'} > \ell_{q_i'}$, we are done. Otherwise, we remove p_i' and q_i' from

p_i and q_i and continue the pruning for the rest. This must terminate in a pair of paths p_i' and q_i' with $\ell_{p_i'} > \ell_{q_i'}$ because at any point in the pruning we keep the property that for the remaining paths p_i and q_i, $\ell_{p_i} > \ell_{q_i}$.
Let the endpoints of the resulting paths p_i' and q_i' be called s and t. Recall that p_i' is along edges e with $\lambda_e(P \setminus Q) > 0$ and q_i' is along edges e with $\lambda_e(Q \setminus P) > 0$. Thus, for each edge e in p_i', $\delta_e > d \cdot T$, which implies that the distance in height between s and t must fulfill

$$h(s) - h(t) > d \cdot T \cdot \ell_{p_i'} \ .$$

On the other hand, it holds for each edge e in q_i' that $\delta_e < d(T+1)$, which implies that the distance in height between s and t must also fulfill

$$h(s) - h(t) < d(T+1) \cdot \ell_{q_i'} \ .$$

Thus,

$$d(T+1) \cdot \ell_{q_i'} \ > \ d \cdot T \cdot \ell_{p_i'} \tag{1}$$

$$\frac{T+1}{T} \ > \ \frac{\ell_{p_i'}}{\ell_{q_i'}} \ . \tag{2}$$

Since $T \geq n$, it follows that $(T+1)/T \leq 1 + 1/n$. On the other hand, $\ell_{p_i'} \leq n-1$ and $\ell_{q_i'} < \ell_{p_i'}$. Hence,

$$\frac{\ell_{p_i'}}{\ell_{q_i'}} \geq \frac{\ell_{p_i'}}{\ell_{p_i'} - 1} \geq \frac{n-1}{n-2} > 1 + \frac{1}{n} \ .$$

This contradicts (2), which proves our claim. □