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Theory of Network Communication

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Solutions to Assignment 1

Problem 1 (4 points)**:**
(a) Show that the modified Bellman-Ford algorithm produces final values for $D_{u,v}(w)$ with $\min_v D_{u,v}(w) = \gamma(u,w)$. (2 points)
Proof. For any two nodes u and w, let $\gamma_\ell(u,w)$ denote the minimum possible maximum edge cost of a path of length at most ℓ from u to w. Obviously, it holds for all $\ell \geq 0$ that

$$\gamma_{\ell+1}(u,w) = \min_v\{\max\{c(u,v), \gamma_\ell(v,w)\}\}$$

for all u and w, where

$$\gamma_0(u,w) = \begin{cases} 0 & u = w \\ +\infty & \text{else} \end{cases}$$

We want to prove by induction on the number of iterations of the while loop that at after the ℓth while loop,

$$\min_v D_{u,v}(w) = \min_v\{\max\{c(u,v), \gamma_\ell(v,w)\}\} \ .$$

For $\ell = 0$ (i.e. before executing the while loop) this is certainly true. So suppose that for $\ell \geq 0$ the hypothesis above has already been shown. Then it holds that after iteration $\ell + 1$,

$$\begin{aligned} D_{u,v}(w) &= \min_v\{\max\{c(u,v), \min_x D_{v,x}(w)\}\} \\ &= \min_v\{\max\{c((u,v)), \min_x\{\max\{c(v,x), \gamma_\ell(x,w)\}\}\}\} \\ &= \min_v\{\max\{c((u,v)), \gamma_{\ell+1}(v,w)\}\} \ . \end{aligned}$$

Hence, after $n-1$ iterations,

$$\min_v D_{u,v}(w) = \min_v\{\max\{c(u,v), \gamma_{n-1}(v,w)\}\} = \gamma_n(u,w) \ .$$

Since we only have edges of nonnegative cost, any path with minimum possible maximum edge cost in a network with n nodes can have a length of at most n. Thus, after $n-1$ iterations, we must have that

$$\min_v D_{u,v}(w) = \gamma(u,w) \ .$$

□

(b) Show that under the assumption that $\min_v D_{u,v}(w) = \gamma(u,w)$, the edge selection rules indeed create an MST. (2 points)
Proof. We have to show three properties for the graph T constructed by the rules:

1. T is connected.
2. T does not contain any cycles.
3. T is a spanning tree of minimum cost.

(1): Suppose that T is not connected. Then there is a cut $(U, \bar{U})$ in the network so that no edge in T crosses the cut. Consider now the cheapest edge across this cut, and let us call this edge $\{u, v\}$. Any path from u to v must cross this cut. Thus, there are only two options for any such path p: p contains e or p contains some other edge f crossing the cut. In both cases, $\gamma(p) \geq c(e)$, and therefore e would have to belong to T, contradicting our assumption that T is disconnected.
(2): Suppose that T contains a cycle, C. Let $\{u, v\}$ be the edge of maximum cost on C. Since we assumed that all edge costs are different, there is a path p from u to v along C with $\gamma(p) < c(u, v)$. Hence, our rules would not include $\{u, v\}$ in T, contradicting our assumption.
(3): Notice that when having edges of different costs, the minimum spanning tree is unique. Every edge $\{u, v\}$ of the MST must be taken by our algorithm because it fulfills the property that $c(u, v)$ is at most the minimum possible maximum edge capacity for all paths from u to v. Adding any further edge would create a cycle that cannot exist due to (2). □

Problem 2 (6 points)**:**
Implement the (original) Bellman-Ford algorithm in the Spheres simulation environment using the following graph.

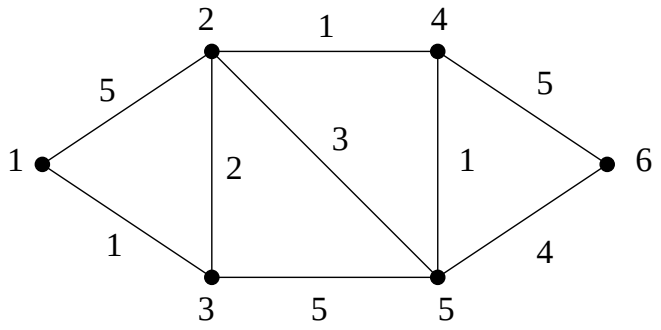

Figure 1: An example graph for the Bellman-Ford algorithm. The numbers at the nodes represent node numbers, and the numbers at the edges represent edge costs.

The final $\min_v D_{u,v}(w)$ values for all u and w are:

$u \backslash w$	1	2	3	4	5	6
1	0	3	1	4	5	9
2	3	0	2	1	2	6
3	1	2	0	3	4	8
4	4	1	3	0	1	5
5	5	2	4	1	0	4
6	9	6	8	5	4	0