

Theory of Network Communication

Fall 2003

Solutions to Assignment 1

Problem 1 (4 points):

(a) Show that the modified Bellman-Ford algorithm produces final values for $D_{u,v}(w)$ with $\min_v D_{u,v}(w) = \gamma(u, w)$. (2 points)

Proof. For any two nodes u and w , let $\gamma_\ell(u, w)$ denote the minimum possible maximum edge cost of a path of length at most ℓ from u to w . Obviously, it holds for all $\ell \geq 0$ that

$$\gamma_{\ell+1}(u, w) = \min_v \{ \max \{ c(u, v), \gamma_\ell(v, w) \} \}$$

for all u and w , where

$$\gamma_0(u, w) = \begin{cases} 0 & u = w \\ +\infty & \text{else} \end{cases}$$

We want to prove by induction on the number of iterations of the while loop that at after the ℓ th while loop,

$$\min_v D_{u,v}(w) = \min_v \{ \max \{ c(u, v), \gamma_\ell(v, w) \} \} .$$

For $\ell = 0$ (i.e. before executing the while loop) this is certainly true. So suppose that for $\ell \geq 0$ the hypothesis above has already been shown. Then it holds that after iteration $\ell + 1$,

$$\begin{aligned} D_{u,v}(w) &= \min_v \{ \max \{ c(u, v), \min_x D_{v,x}(w) \} \} \\ &= \min_v \{ \max \{ c(u, v), \min_x \{ \max \{ c(v, x), \gamma_\ell(x, w) \} \} \} \} \\ &= \min_v \{ \max \{ c(u, v), \gamma_{\ell+1}(v, w) \} \} . \end{aligned}$$

Hence, after $n - 1$ iterations,

$$\min_v D_{u,v}(w) = \min_v \{ \max \{ c(u, v), \gamma_{n-1}(v, w) \} \} = \gamma_n(u, w) .$$

Since we only have edges of nonnegative cost, any path with minimum possible maximum edge cost in a network with n nodes can have a length of at most n . Thus, after $n - 1$ iterations, we must have that

$$\min_v D_{u,v}(w) = \gamma(u, w) .$$

□

(b) Show that under the assumption that $\min_v D_{u,v}(w) = \gamma(u, w)$, the edge selection rules indeed create an MST. (2 points)

Proof. We have to show three properties for the graph T constructed by the rules:

1. T is connected.
2. T does not contain any cycles.
3. T is a spanning tree of minimum cost.

(1): Suppose that T is not connected. Then there is a cut (U, \bar{U}) in the network so that no edge in T crosses the cut. Consider now the cheapest edge across this cut, and let us call this edge $\{u, v\}$. Any path from u to v must cross this cut. Thus, there are only two options for any such path p : p contains e or p contains some other edge f crossing the cut. In both cases, $\gamma(p) \geq c(e)$, and therefore e would have to belong to T , contradicting our assumption that T is disconnected.

(2): Suppose that T contains a cycle, C . Let $\{u, v\}$ be the edge of maximum cost on C . Since we assumed that all edge costs are different, there is a path p from u to v along C with $\gamma(p) < c(u, v)$. Hence, our rules would not include $\{u, v\}$ in T , contradicting our assumption.

(3): Notice that when having edges of different costs, the minimum spanning tree is unique. Every edge $\{u, v\}$ of the MST must be taken by our algorithm because it fulfills the property that $c(u, v)$ is at most the minimum possible maximum edge capacity for all paths from u to v . Adding any further edge would create a cycle that cannot exist due to (2). \square

Problem 2 (6 points):

Implement the (original) Bellman-Ford algorithm in the Spheres simulation environment using the following graph.

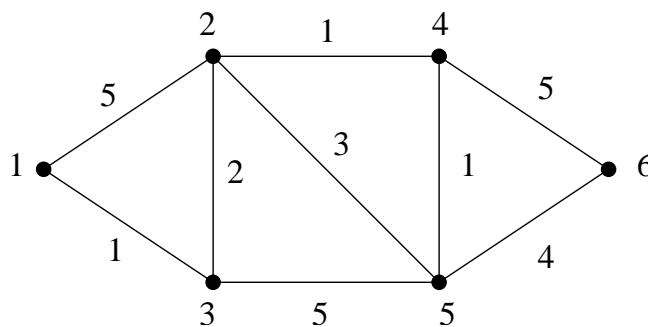


Figure 1: An example graph for the Bellman-Ford algorithm. The numbers at the nodes represent node numbers, and the numbers at the edges represent edge costs.

The final $\min_v D_{u,v}(w)$ values for all u and w are:

$u \backslash w$	1	2	3	4	5	6
1	0	3	1	4	5	9
2	3	0	2	1	2	6
3	1	2	0	3	4	8
4	4	1	3	0	1	5
5	5	2	4	1	0	4
6	9	6	8	5	4	0