7 Scheduling

So far, we mostly looked at *static* routing problems such as BMFPs, i.e. all demands (or packets) are given from the beginning and no new demands (or packets) arrive during the routing. In the real world, however, packets are usually injected at the nodes in a continuous fashion. Models that take this into account are called *dynamic routing models*, as opposed to the *static routing models* we mostly considered before. These models can be usually characterized as either *stochastic* or *adversarial*. In stochastic models, the injection of packets is modelled with the help of stochastic processes, whereas in the adversarial model it is assumed that an adversary controls the injection of new packets [2]. In this section, we restrict ourselves to considering the following variant of the adversarial model, which is due to [1].

7.1 The adversarial injection model

In this model we have an adversary that is allowed to demand network bandwidth up to some specified limit. That is, it is allowed to choose any node at any time step to inject a new packet and it is allowed to select any path for the injected packet as long as the load of the edges does not exceed a certain limit. More formally, for any $w, \lambda > 0$, an adversary is called a (w, λ)-*bounded adversary* if for all edges e and all time intervals I of length w, it injects no more than $\lambda \cdot w$ packets during I that contain edge e in their routing paths. λ is called the *injection rate* and w is called the *burstiness* of the adversary. We demand that the adversary has to tell the system the paths it selects. Thus, we are left with solving a *dynamic scheduling problem*. An algorithm for this problem is called a *scheduling protocol*. A scheduling protocol is called *stable* for some λ and some network G if for any injection rate of at most λ and any paths selected for the packets in G the (expected) worst-case routing time of a packet (i.e. the time it spends in the system) does not grow unboundedly with time. Since an edge can transport at most one packet per step, λ can be at most 1. A protocol that is stable for any network and any $\lambda < 1$ is called *universally stable*. We will show that there are both universally stable and non-universally stable protocols.

7.2 Queueing disciplines

Usually, every node has a packet queue for every outgoing edge. Hence, the simplest form of a scheduling protocol are protocols in which we just define a rule about how to rank packets in a queue so that we know which packet to send out next along an edge. Elementary queueing disciplines are:

- FIFO (first in first out): gives preference to the packet that was the first to arrive at the queue (among those that are still in the queue).
- NTO (nearest to origin): gives preference to the packet that has traveled the smallest amount of edges so far. Ties are broken arbitrarily.
- FTG (furthest to go): gives preference to the packet that has the largest number of edges to go. Ties are broken arbitrarily.
- NTG (nearest to go): gives preference to the packet that has the smallest number of edges to go. Ties are broken arbitrarily.

- SIS (shortest in system): gives preference to the youngest packet.
- LIS (longest in system): gives preference to the oldest packet.

Due to its simplicity, FIFO is the most widely used scheduling protocol in practice.

7.3 Universal stability of SIS

The SIS protocol always gives preference to the youngest packet. Ties may be broken arbitrarily. For this protocol we can prove the following amazing result.

Theorem 7.1 *SIS is universally stable.*

Proof. Let $0 < \epsilon < 1$ be chosen such that $\lambda = 1 - \epsilon$. Suppose that there is a packet P that requires more than $\sum_{i=1}^{d}(w+1)/\epsilon^i$ steps to traverse a path of length d. In this case there must be an $i \in \{1, \ldots, d\}$ for which P was delayed for at least $(w+1)/\epsilon^i$ steps at the ith edge of its path. Consider the minimal i for which this holds. Let e be the corresponding edge on P's path. Then the time difference T between the injection of P and the time at which P had to wait for the $(w+1)/\epsilon^i$th time at e is at most

$$\sum_{j=1}^{i}(w+1)/\epsilon^j = \frac{w+1}{\epsilon} \cdot \frac{(1/\epsilon)^i - 1}{(1/\epsilon) - 1} = (w+1) \cdot \frac{(1/\epsilon)^i - 1}{1-\epsilon} \; .$$

Since $\lambda = 1 - \epsilon$, at most

$$(1-\epsilon)T + w = (w+1) \cdot \Big((1/\epsilon)^i - 1\Big) + w < (w+1)/\epsilon^i$$

packets can be injected during T time steps that intend to cross e. Only these packets can be preferred against P at e. However, since our assumption requires P to be delayed by at least $(w+1)/\epsilon^i$ packets at e, we arrive at a contradiction Thus, the delay of a packet following a path of length d can be at most $\sum_{i=1}^{d}(w+1)/\epsilon^i$, and therefore SIS must be stable. □

7.4 Universal stability of LIS

In contrast to SIS, the LIS protocol always gives preference to the oldest packet. Also LIS is stable.

Theorem 7.2 *LIS is universally stable.*

Proof. Let $\lambda = 1 - \epsilon$ and let D be the maximal length of a path that can be chosen by the adversary. Suppose that there is a packet that requires more than

$$\sum_{j=0}^{d-1}(w+1)/\epsilon^{D-j}$$

steps to traverse a path of length d. Let P be the first packet with this property and let its ith edge be the first edge at which this property is fulfilled. Let e be the ith edge on P's path, and let t be chosen such that at the beginning of time step t P has an age of $\sum_{j=0}^{i-2}(w+1)/\epsilon^{D-j}$ steps. Then P was delayed

at e for at least $(w+1)/\epsilon^{D-(i-1)}$ steps following t. Since, according to our assumptions, the maximal age of any packet before time step $t+(w+1)/\epsilon^{D-(i-1)}$ was at most $\sum_{j=0}^{D-1}(w+1)/\epsilon^{D-j}$, the difference T between this age and the age of P at time step t is at most

$$\begin{aligned}\sum_{j=0}^{D-1}(w+1)/\epsilon^{D-j} - \sum_{j=0}^{i-2}(w+1)/\epsilon^{D-j} &= \sum_{j=i-1}^{D-1}(w+1)/\epsilon^{D-j} = \sum_{j=1}^{D-i+1}(w+1)/\epsilon^{j} \\ &= \frac{w+1}{\epsilon}\cdot\frac{(1/\epsilon)^{D-i+1}-1}{(1/\epsilon)-1} \\ &= (w+1)\cdot\frac{(1/\epsilon)^{D-i+1}-1}{1-\epsilon}\,.\end{aligned}$$

Since $\lambda = 1-\epsilon$, at most

$$(1-\epsilon)T + w = (w+1)\left((1/\epsilon)^{D-i+1}-1\right) + w < (w+1)/\epsilon^{D-i+1}$$

packets can be injected from time step $t - \sum_{j=0}^{D-1}(w+1)/\epsilon^{D-j}$ to the injection time of P that intend to cross e. Since these are the only packets that can delay P at e, and our assumptions require P to be delayed by at least $(w+1)/\epsilon^{D-i}$ packets, we arrive at a contradiction. Hence, LIS must be stable. □

7.5 Instability of FIFO

The FIFO protocol always gives preference to the packet that was the first to arrive at a node. The next result demonstrates that FIFO is not universally stable.

Theorem 7.3 *For $\lambda \geq 0.85$ there is a network and an adversary that causes FIFO to be unstable.*

Proof. The adversary uses the network shown in Figure 1. It is not difficult to check that this network can be embedded in a grid and a 3-dimensional hypercube. Thus, it is possible for our counterexample to happen in important standard networks.

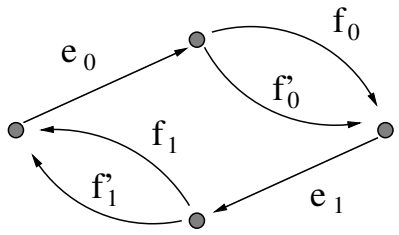

Figure 1: The counterexample for FIFO.

We divide the injection strategy of the adversary into several phases. Our induction hypothesis will be that at the beginning of phase j there are at least $s+j$ packets in the buffer of e_i that intend to cross the edges e_i and f_i, where $i = f \bmod 2$ and s is a sufficiently large constant.

In phase 1 we have to inject packets in such a way that at the end of this phase there are $m = s+2$ packets in the buffer of e_0 that have a remaining path of $(e_0 f_0)$. This can be achieved by attaching a set of m lines of nodes to the starting node of e_0, where line i has a length of i/λ. An injection rate of

λ allows m packets to be injected at the endpoints of the lines, one per endpoint, so that all m packets reach e_0 at the same time. This ensures the induction hypothesis for phase 2.

We consider now an arbitrary even phase j. The odd phases work in the same way. We will show that if at the beginning of phase j the buffer of e_0 contains a set M_0 of $m = s + j$ packets with remaining path $(e_0 f_0)$, then at the beginning of phase $j + 1$ there will be at least $m + 1$ packets in the buffer of e_1 that have a remaining path of $(e_1 f_1)$.

In the following we describe the injection strategy of the adversary for phase j. To simplify the proof, we will avoid dealing with floors and ceilings. If m is sufficiently large, then their effects can be neglected. Phase j consists of 3 stages.

Stage 1 consists of m steps. During this stage we inject a set M_1 of λm packets with path $(e_0 f'_0 e_1 f_1)$. These packets are blocked by the packets in M_0. Furthermore, we inject λm packets with path (f_0). These packets ensure that at the end of stage 1 at least λm packets with remaining path (f_0) are still in the buffer of f_0.

Stage 2 consists of λm steps. During this stage we inject a set M_2 of $\lambda^2 m$ packets with path $(f_0 e_1 f_1)$. These are blocked by the packets that are already in the buffer of f_0. Furthermore, we inject $\lambda^2 m$ packets with path (f'_0). These mix with the packets in M_1. Every $1/\lambda$ steps we have in addition to $1/\lambda$ packets from M_1 that reach the buffer of f'_0 one packet with path (f'_0). This has the effect that every $1 + 1/\lambda$ steps $1/\lambda$ packets from M_1 pass f'_0, which reduces the number of packets in the buffer of f'_0 after stage 2 to

$$\lambda m - \frac{\lambda m}{1 + 1/\lambda} \cdot \frac{1}{\lambda} = \lambda m - \frac{\lambda m}{\lambda + 1} = \frac{\lambda^2 m}{\lambda + 1} .$$

Stage 3 consists of $\lambda^2 m$ steps. During this stage, the packets in M_1 and M_2 move forward and mix in the buffer of e_1. Furthermore, $\lambda^3 m$ packets are injected with path $(e_1 f_1)$. Since $\lambda^2 m$ packets traverse e_1, the number of packets in the buffer of e_1 with remaining path $(e_1 f_1)$ after stage 3 is equal to $\lambda^3 m + \lambda^2 m/(\lambda + 1)$.

This ends phase j. Since for $\lambda \geq 0.85$ we have that $\lambda^3 + \lambda^2/(\lambda + 1) > 1$, we arrive at the induction hypothesis for phase $j + 1$. □

Although FIFO cannot handle well adversarial traffic, it was shown by Bramson that it is actually universally stable for stochastic traffic [3]. Since FIFO queues are so much easier to build than other queues and Internet traffic is of somewhat stochastic nature, this explains why there has been no need so far to choose queueing disciplines other than FIFO in Internet routers.

7.6 Low delay bounds

The stability results for SIS and LIS above demonstrate that the worst-case delay of a packet is finite, but it may be very large. In this section we demonstrate that in certain situations much better delay bounds can be shown.

Consider using the LIS rule in leveled networks. We assume that every packet has a rank that is determined by its birth date plus some small offset < 1 that gives the packets a strict ordering (so that we never run into a tie). When using the adversary in a leveled network, we only allow the adversary to inject paths that go from lower to higher levels, i.e. for every edge e on such a path, e must go from some node in level k to some node in level $k + 1$.

Theorem 7.4 *Let L denote the depth of the leveled network. For any (w, λ)-bounded adversary with $\lambda \leq 1$, LIS is stable, and every packet reaches its destination in at most $(1 + \lambda w)L$ time steps.*

Proof. Suppose on the contrary that there is a packet p_0 that needs more than $(1 + \lambda w)L$ time steps to reach its destination. We will use a delay sequence argument to show that this is not possible. First, we follow p_0 backwards in time from the point where it reached its destination until it was delayed by some packet p_1. We then follow p_1 backwards in time until is was delayed by some packet p_2, and so on, until we get to a packet p_s that had no prior delays. The path q recorded in this process is called a *delay path* and must have a length of at most L, because we only allow the adversary to inject paths with edges from level k to $k + 1$ for some k. Since LIS is used, we also know that for every $i \in \{1, \ldots, s\}$, p_i must be at least as old as p_{i-1}. Hence, the time spanned by the delay sequence must be more than

$$(\text{birth}(p_0) - \text{birth}(p_s)) + (1 + \lambda w)L \ ,$$

and the total amount of delay events must therefore be more than

$$(\text{birth}(p_0) - \text{birth}(p_s)) + \lambda w L \ .$$

Furthermore, for every edge e along the delay path q we can associate an interval I_e containing the birth dates of all packets recorded in the delay sequence at e. Since the birth dates of the packets are strictly decreasing, the I_e's can be chosen so that

- $\sum_{e \in q} |I_e| \leq \text{birth}(p_0) - \text{birth}(p_s)$, where
- for every edge e at which no delay was recorded, $I_e = [r, r]$ where r is the rank of the packet that passed through e, and
- for any two edges e and e' on the delay path q, $I_e \cap I'_e = \emptyset$.

On the other hand, the number of packets that can be injected during I_e with a path through e is at most $\lambda(|I_e| + w)$. Hence, the number of packets that can be injected during the time span of the delay sequence, $\bigcup_e I_e$, is at most

$$\sum_{e \in q} \lambda(|I_e| + w) \leq \sum_{e \in q} |I_e| + \lambda w L \leq (\text{birth}(p_0) - \text{birth}(p_s)) + \lambda w L \ ,$$

which contradicts the fact that there have to be more than $(\text{birth}(p_0) - \text{birth}(p_s)) + \lambda w L$ delays along the delay path. □

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