Theory of Network Communication

Fall 2002

Solutions to Assignment 1

Problem 1:

Most responses to this were accepted. A common answer was that the edge expansion provides a bound on the flow number which by definition 1.09 and theorem 1.10 is a good indicator of the possible congestion and dilation in the network. Also accepted was the intuitive explanation that the edge expansion identifies bottlenecks in the network which would limit routing performance.

Problem 2:

For uniform edge capacities, assume individual edge capacities are 1 in each direction. The edge expansion is $\alpha = \frac{2}{n}$, found by selecting one half of the cycle to be in U (thus $c(U,\overline{U}) = 2$, $c(U) = c(\overline{U}) = n$). The flow number can be found by considering the BMFP/PMFP problem mentioned in the lecture notes, where there is a demand of $\frac{c(v) \cdot c(w)}{c(V)}\ \forall\ v, w \in V$. Then by our assumption demand is $\frac{2 \cdot 2}{2n} = \frac{2}{n}$.

The shortest-paths system implies that we send the flows along the shortest paths to their destination. Each node then will send a 1-edge long flow in each direction to its neighbors, and a 2-edge long flow in each direction to its neighbors' neighbors, and so on. For simplicity let us send the $n/2$-edge flow along both directions, and remove that later. Then the total edge capacities requested by flows sent from a single vertex are:

$$2 \cdot \left(\frac{2}{n} \sum_{i=1}^{n/2} i \right) - \left(\frac{2}{n} \cdot \frac{n}{2} \right) = \frac{4}{n} \cdot \frac{(n/2)(n/2+1)}{2} - 1 = \frac{n}{2} + 1 - 1 = \frac{n}{2}$$

The two is multiplied by the summation to handle both directions that flow is sent. The summation is over each possible length of a flow to indicate the total capacity required by each flow, and the $\frac{2}{n}$ outside the summation represents the demand for the flow. The subtracted term is cancelling out one of the $\frac{n}{2}$-length flows (since we counted it twice previously). Thus the total capacity required by the flows sent by one node is $\frac{n}{2}$. Multiply this by n to get a total capacity demanded of $\frac{n^2}{2}$. But the graph is entirely symmetric; therefore the capacity demanded of each edge is $\frac{n}{2}$, which is the congestion (what does the current edge capacity of 1 need to be multiplied by to be sufficient to handle all demand). Since the dilation is also $\frac{n}{2}$, the flow number is $\frac{n}{2}$, which is equal to the inverse of the edge expansion.

Problem 3:

The min-cut ratio is $\frac{1}{3}$, which is realized by cutting only the middle edge (between s_2 and t_2) - the cut has a capacity of 1 but a demand of 3 (flows 2, 4, and 5). This is achievable by setting a flow of $\frac{1}{3}$ for all flows... this clearly is under capacity. Since we have a flow equal to a cut amount, both are optimal by the theorems in the lecture notes.

The minimum multicut is achieved by cutting edges e_1, e_3, e_5, for capacity 3. A maximum throughput can be achieved by setting a flow of 1 for commodities 1, 2, and 3, and 0 for 4 and 5. This then has throughput 3, which is equal to the multicut capacity, again proving that both are optimal.