SPHERICAL WIENER FILTER

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ABSTRACT

A novel group-theoretic method is presented for denoising a three-dimensional scene in isotropic noise. Images of the scene at varying depths are regarded as reference stochastic processes on the unit sphere to formulate Weiner-Hopf equations for estimating the image at any given depth. These comprise a set of coupled linear integral equations on the unit sphere and are solved using Peter-Weyl theory of Fourier transform on the rotation group. The computational complexity of this algorithm is reduced using bi-invariance of the image correlations with respect to the stabilizer subgroup of the rotation group.

***Index Terms*—** Wiener filtering, smoothing methods, 3D surface data, spherical harmonics.

1. INTRODUCTION

With the advances in modern acquisition devices large volume of experimental data on high dimensional nontrivial manifolds is readily available. The analysis and processing of such complex data requires new and sophisticated signal processing techniques. In astrophysics and cosmology, the cosmic microwave background (CMB) radiation data is collected in all directions of the sky on a sphere. In geophysics, remote sensing of the Earth's surface and atmosphere generates spherical data maps which are crucial for understanding climate change, geodynamics or monitoring human-environment interactions. In robotics, omnidirectional cameras capture the 3D scene on a hyperbolic mirror which can be mapped onto a regular spherical grid. Various signal processing tasks on these spherical images are realized by the application of non-commutative spherical harmonic analysis techniques [1].

In many settings, however, the data is available on arbitrary manifolds. In biomedical imaging and computer vision, three dimensional surface data is acquired with range scanners or stereovision systems. If the data corresponds to *star-shaped* objects (objects with surfaces topologically isomorphic to the sphere), then the data may be considered to be a height field on the sphere [2]. The other approach is to use triangular meshes to model such complex shapes and employ clever parametrization to transform the surface mesh into a spherical signal [3].

Inspired by a multitude of applications of signal processing on the sphere, a Wiener filter was derived for isotropic signal fields in [4]. This finds application in denoising of images as well as in reconstruction from a sparse set of samples. Few researchers have previously employed spherical diffusion techniques [2] or filtering [3] for smoothing and denoising 3-D surface data but the filters used were smooth Gaussian kernels with low pass characteristics. Wiener filtering results in better performance since it is optimal in the minimum mean squared error sense. It thus results in better reconstruction and denoising for omni-directional images and 3-D surface data that can be represented as functions (or images) on the sphere.

This paper presents an efficient Wiener filter. The computational complexity of the algorithm is reduced by using bi-invariance of the image correlations with respect to the stabilizer subgroup of the rotation group. The spherical Wiener filtering problem is formulated in Section 2.2 and Wiener-Hopf equations are derived on a homogeneous space. The theory is developed in complete generality for functions defined on a manifold M under the transitive action of a compact group G. Wiener-Hopf equations on the unit sphere comprise a set of coupled linear integral equations which are expressed as convolutional integral equations on the rotation group. The integral equations are expressed in Fourier domain as discrete sums over the irreducible representations of the rotation group using the Peter-Weyl theory [5]. The complexity reduction is discussed in Section 3.2 and specialized in Section 3.3 to the case where the manifold is the sphere S^2 and the group is the three dimensional rotation group [6]. Experimental results are presented in Section 4.

2. PROBLEM FORMULATION

Inspired by the group theoretical methods in image processing as outlined in [5, 7] the problem is addressed here using the non-commutative harmonic analysis based on the irreducible representations of the group G.

2.1. Mathematical Preliminaries

This section introduces the reader to the Peter-Weyl theorem which is at the heart of the analysis in this paper. A basic yet useful measure-theoretic result is also presented.

2.1.1. *Peter-Weyl theory*

Let G be a compact Lie group acting transitively on a manifold M. A representation of G on a vector space V_π is a group homomorphism π from G to the group of all bijective linear transformations that map V_π to itself. The dimension of the representation π is the dimension d_π of the representation space V_π. For each $g \in G$, $\pi(g)$ can be thought of as a $d_\pi \times d_\pi$ invertible matrix operating on V_π. Let $\hat{G}$ denote the set of all inequivalent irreducible unitary representations of G. This implies that all matrices in $\hat{G}$ are unitary, no matrix can be expressed as direct sum of others and no two matrices are similar. The Peter-Weyl theorem states that $\{\sqrt{d_\pi}\pi_{\alpha,\beta}(g)|1 \leq \alpha, \beta \leq d_\pi, \pi \in \hat{G}\}$ is a complete orthonormal basis for $L^2(G)$, the space of all square integrable functions on G. Thus, for any $f \in L^2(G)$,

$$f(g) = \sum_{\pi \in \hat{G}} \sum_{1 \leq \alpha,\beta \leq d_\pi} d_\pi \langle f, \pi_{\alpha\beta} \rangle \pi_{\alpha\beta}(g) \tag{1}$$

where $\langle f, \pi_{\alpha\beta} \rangle = \int_G f(g)\overline{\pi_{\alpha\beta}(g)}dg$. In compact notation the $d_\pi \times d_\pi$ matrix $\hat{f}(\pi) = \int_G f(g)\pi^*(g)dg$ is defined to be the *Fourier transform* of f at frequency $\pi \in \hat{G}$. Similarly, equation (1) can be expressed compactly as

$$f(g) = \sum_{\pi \in \hat{G}} d_\pi \mathrm{Tr}\left(\hat{f}(\pi)\pi(g)\right) \tag{2}$$

where $\mathrm{Tr}(\mathbf{A})$ represents the trace of the matrix $\mathbf{A}$. Equation (2) defines the *inverse Fourier transform* of $\hat{f}$ at g.

2.1.2. *Change of measure formula*

Often it is convenient to make a variable substitution so that a function defined on a manifold can be expressed as a function on a group acting on the manifold. This allows signals defined on the manifold to be expanded in the Fourier series (2). Also the integrals on the manifold can be expressed as integrals on the group. The following elementary but useful measure theory result allows for such substitutions.

Let x_0 be a fixed point on the manifold M. Consider a measurable map τ from measure space $(G, \mathcal{F}_G, \mu)$ to measure space $(M, \mathcal{F}_M, \nu)$ given by $\tau(g) = g \cdot x_0$. Here μ is the natural Haar measure associated with the compact group G and ν is a G-invariant measure on M induced by τ (define $\nu(E) = \mu(\tau^{-1}(E)) \forall E \in \mathcal{F}_M$). Therefore, if f is a measurable function on M that is integrable w.r.t. ν then $f \circ \tau$ is integrable w.r.t. μ and

$$\int_M f(x)\nu(dx) = \int_G f(g \cdot x_0)\mu(dg) \tag{3}$$

Also important is the measurable cross-section map $\gamma : M \to G$ which for every $x \in M$ gives $\gamma(x) \in G$ such that $\gamma(x) \cdot x_0 = x$. This allows for the reverse substitution: if f is a square integrable function on M then it can be developed into a Fourier series over G,

$$f(x) = \sum_{\pi \in \hat{G}} d_\pi \mathrm{Tr}(\hat{f}(\pi)\pi(\gamma(x))) \tag{4}$$

2.2. Wiener Hopf equations on homogeneous space

Let G be a compact Lie group acting transitively on a manifold M. Suppose that $\{\eta_1(x), \ldots, \eta_k(x), \xi(x)|x \in M\}$ is a collection of $k+1$ real or complex-valued random processes. The variables η_i denote the k observations or samples on the manifold and ξ denotes the unobserved signal that needs to be reconstructed based on the samples. The random processes are assumed to be zero mean, $\mathbb{E}[\xi(x)] = 0$, $\mathbb{E}[\eta_i(x)] = 0$, and jointly wide sense G-stationary, i.e. the correlation functions satisfy

$$\begin{aligned} \mathbb{E}[\xi(x)\overline{\xi(y)}] &= R(x,y) = R(gx,gy) \\ \mathbb{E}[\xi(x)\overline{\eta_j(y)}] &= R_j(x,y) = R_j(gx,gy) \\ \mathbb{E}[\eta_i(x)\overline{\eta_j(y)}] &= R_{ij}(x,y) = R_{ij}(gx,gy) \end{aligned} \tag{5}$$

for all $g \in G$, $x, y \in M$ and $i, j = 1, \ldots, k$. The best linear estimate $\hat{\xi}(x)$ of the process $\xi(x)$ based on η_i's is

$$\hat{\xi}(x) = \sum_{i=1}^{k} \int_M L_i(x,y)\eta_i(y)dy \tag{6}$$

where integration is with respect to the unique G-invariant probability measure on the compact manifold M and $L_i(x,y)$ are unknown functions to be determined. To ensure that $\hat{\xi}(x)$ is G-stationary process, $L_i(x,y)$ should be G-invariant, i.e. $L_i(gx,gy) = L_i(x,y)$ for $g \in G,\ x, y \in M$.

By the orthogonality principle, the mean square error $\mathbb{E}|\xi(x) - \hat{\xi}(x)|^2$ is minimized when $\mathbb{E}[(\xi(x) - \hat{\xi}(x))\overline{\eta_j(y)}] = 0$ for $j = 1, \ldots, k$ and $x, y \in M$. Using equation (6) and definitions in (5), the orthogonality principle gives the following normal equations

$$R_j(x,y) = \sum_{i=1}^{k} \int_M L_i(x,z)R_{ij}(z,y)dz \tag{7}$$

for $1 \leq j \leq k$ and $x, y \in M$. We need to find G-invariant functions L_i's that satisfy these coupled linear integral equations on the manifold. These equations can be expressed as relations on the group G as follows.

Fix an 'origin' x_0 in M. Then, by the transitivity of the group action, there exist $g_1, g_2 \in G$ such that $x = g_1 x_0$ and $y = g_2 x_0$. Then (7) can be expressed as

$$R_j(g_1x_0, g_2x_0) = \sum_{i=1}^{k} \int_M L_i(g_1x_0, z)R_{ij}(z, g_2x_0)dz. \tag{8}$$

Using the fact that the map $g \to gx_0$ from G into M takes the normalised Haar measure of G to the unique G-invariant

probability measure on M, the integral in equation above can be transformed into an integral on G, giving

$$R_j(g_1x_0, g_2x_0) = \sum_{i=1}^{k} \int_G L_i(g_1x_0, gx_0) R_{ij}(gx_0, g_2x_0) dg$$

for $g_1, g_2 \in G$ where dg indicates integration with respect to the Haar measure on G. It is convenient to use different notation when viewing the correlation functions as functions on the group. Lets define $\phi_j(g) = R_j(x_0, gx_0)$, $\psi_{ij}(g) = R_{ij}(x_0, gx_0)$ and $l_i(g) = L_i(x_0, gx_0)$. Then using the invariance conditions it is easily verified that for all $g \in G$,

$$\phi_j(g) = \phi_j(g_1^{-1}g_2) = \sum_{i=1}^{k} \int_G l_i(g')\psi_{ij}(g'^{-1}g)dg'. \quad (9)$$

The convolutional integral equations in (9) are solved using the Peter-Weyl theory on the compact group G [7].

3. SPHERICAL WIENER FILTER

3.1. Solution using irreducible representations

Taking the Fourier transform on both sides of (9) yields

Lemma 1. $\hat{\phi}_j(\pi) = \sum_{i=1}^{k} \hat{l}_i(\pi)\, \hat{\psi}_{ij}(\pi)$, *for* $j = 1, \ldots, k$.

This gives a system of linear equations which can be solved for the unknown $\hat{l}_i(\pi)$. The filter responses $L_i(x, y)$ can then be obtained as follows. Since G is transitive on M, construct a measurable cross-section $\gamma : M \to G$ as discussed in Section 2.1.2. Then

$$\begin{aligned} L_i(x, y) &= L_i(\gamma(x)x_0, \gamma(y)x_0) \\ &= L_i(x_0, \gamma^{-1}(x)\gamma(y)x_0) = l_i(\gamma^{-1}(x)\gamma(y)) \quad (10) \end{aligned}$$

for $1 \le i \le k$ and $x, y \in M$, where l_i is the inverse Fourier transform of $\hat{l}_i$. The functions $\{L_i(x, y)\}_{i=1}^{k}$ comprise the Wiener filter and can be used in equation (6) to obtain the estimates $\hat{\xi}$. However, each of the equations in Lemma 1 involves $d_\pi \times d_\pi$ matrices and is very intensive computationally.

3.2. Complexity reduction

An important result is presented in this section which helps reducing the computational complexity. Consider the transformations that leave the origin x_0 fixed, the *stabilizer* of x_0, $H = \{h \in G : hx_0 = x_0\}$. H is a closed subgroup of G and has its own Haar measure μ_H. A function f on G is said to be H*-bi-invariant* if $f(h_1gh_2) = f(g)$ for all $g \in G$ and $h_1, h_2 \in H$. We have the following result for such functions.

Theorem 1. *The Fourier transform of an H bi-invariant function f satisfies $P_\pi \hat{f}(\pi) P_\pi = \hat{f}(\pi)$ for every $\pi \in \hat{G}$, where*

$$P_\pi = \int_H \pi(h)\mu_H(dh) \quad (11)$$

is the orthogonal projection onto

$$W_\pi = \{x \in V_\pi : \pi(h)x = x \;\forall\; h \in H\}. \quad (12)$$

Proof. Omitted. □

The significance of this result is that the Fourier transform matrix $\hat{f}(\pi)$, for an H bi-invariant function f, lies in a smaller dimension subspace: If d_π^0 denotes the dimension of W_π, then the Fourier transform matrix lies in a $d_\pi^0 \times d_\pi^0$ dimensional subspace. We will denote the restriction of the operator $\hat{f}(\pi)$ to the subspace W_π by $\tilde{f}(\pi)$ and restriction of $P_\pi \pi(g) P_\pi$ to the subspace W_π by $\tilde{\pi}(g)$. Then, $\tilde{f}(\pi)$ and $\tilde{\pi}(g)$ can be thought of as $d_\pi^0 \times d_\pi^0$ matrices in any orthonormal basis of the subspace W_π. Using Theorem 1, any square integrable H-bi-invariant function f on G can therefore be recovered from the operators $\{\tilde{f}(\pi), \pi \in \hat{G}\}$ by

$$f(g) = \sum_{\pi \in \hat{G}} d_\pi \mathrm{Tr}\left(\tilde{f}(\pi)\tilde{\pi}(g)\right) \quad (13)$$

Note that the functions ϕ_j, ψ_{ij} and l_i are H bi-invariant and the corresponding Fourier coefficient matrices are $d_{\pi_0} \times d_{\pi_0}$ dimensional in the orthonormal basis for W_π. Consequently, the system of linear equations in Lemma 1 reduces to

$$\tilde{\phi}_j(\pi) = \sum_{i=1}^{k} \tilde{l}_i(\pi)\tilde{\psi}_{ij}(\pi), \quad 1 \le j \le k \text{ and } \pi \in \hat{G} \quad (14)$$

and from (13) Fourier inversion for l_i reduces to

$$l_i(g) = \sum_{\pi \in \hat{G}} d_\pi \mathrm{Tr}\left(\tilde{l}_i(\pi)\tilde{\pi}(g)\right), \quad 1 \le i \le k. \quad (15)$$

Typically, d_{π_0} is much smaller than d_π.

3.3. The case $G = SO(3)$, $M = S^2$

The Wiener filter is now specialized to the action of the rotation group $G = SO(3)$ on the unit sphere $M = S^2$. The irreducible representations of the rotation group are discussed in [4, 5]. The set of irreducible inequivalent representations of $SO(3)$ is indexed by positive integers, with π_n denoting the $2n+1$ dimensional representation, for $n \in \{0, 1, 2, \ldots\}$. The fixed point x_0 on the sphere is chosen to be the north pole. The stabilizer subgroup H then corresponds to all rotations of the sphere about the Z-axis and the subspace W_n of all vectors fixed by H is one-dimensional space spanned by the Z-axis. This holds for every π_n and thus $d_{\pi_n}^0 = 1$ for all n. Each of $\tilde{\phi}_j, \tilde{\psi}_{ij}$ and $\tilde{l}_i$ is therefore a scalar and equations (14) and (15) are much easier to solve than the system of equations in Lemma 1. These quantities are computed using the spherical harmonics as discussed below.

Let $\mathbf{x} = [x_1\ x_2\ x_3]'$ be a point on the unit sphere and let $g = \gamma(\mathbf{x})$ be the rotation that rotates north pole to point $\mathbf{x}$. The irreducible representation, $\tilde{\pi}_n$, corresponding to this rotation is given as (see [8]),

$$\tilde{\pi}_n(g) = \tilde{\pi}_n(\gamma(\mathbf{x})) = \sum_{r=0}^{n} \binom{n}{r}^2 \left(\frac{1+x_3}{2}\right)^r \left(\frac{-1+x_3}{2}\right)^{(n-r)} \quad (16)$$

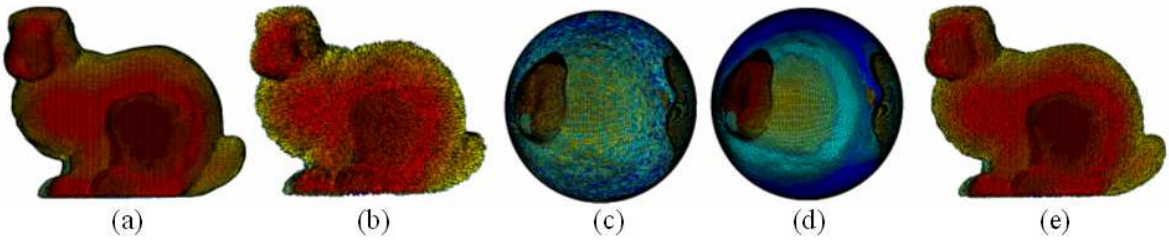

Fig. 1. (a) 3-D bunny image (b) Noisy bunny image (c) Projection of noisy surface data of 3-D bunny on the unit sphere (d) Wiener filtered spherical projection (e) Restored 3-D bunny image.

Define functions ρ_n on the unit sphere S^2 as $\rho_n(\mathbf{x}) = \tilde{\pi}_n(\gamma(\mathbf{x}))$. The functions ρ_n are the *spherical harmonics* and the *spherical harmonic coefficients* for $\phi_j(g) = R_j(\mathbf{x}_0, g\mathbf{x}_0)$ at the frequency $\pi = \pi_n$ is computed as

$$\tilde{\phi}_j(\pi_n) = \int_G R_j(\mathbf{x}_0, g\mathbf{x}_0)\overline{\tilde{\pi}_n(g)}dg = \int_{S^2} R_j(\mathbf{x}_0, \mathbf{x})\rho_n(\mathbf{x})d\sigma(\mathbf{x}) \quad (17)$$

where $d\sigma$ denotes integration with respect to the normalized area measure on S^2, $d\sigma(\mathbf{x}) = \frac{dx_1 dx_2}{|x_3|}$. Similarly,

$$\tilde{\psi}_{ij}(\pi_n) = \int_{S^2} R_{ij}(\mathbf{x}_0, \mathbf{x})\rho_n(\mathbf{x})d\sigma(\mathbf{x}) \quad (18)$$

Recall that R_j and R_{ij} are the correlation functions and the normal equations in Lemma 1 are specialized to

$$\tilde{\phi}_j(\pi_n) = \sum_{i=1}^{k} \tilde{l}_i(\pi_n)\tilde{\psi}_{ij}(\pi_n) \quad (19)$$

where $\tilde{\pi}_n$, $\tilde{\phi}_j$ and $\tilde{\psi}_{ij}$ are given by equations (16), (17) and (18) respectively. The normal equations (19) are solved for $\tilde{l}_i(\pi_n)$ and the inverse Fourier transform of $\tilde{l}_i(\pi_n)$ gives

$$L_i(x, y) = \sum_{n=0}^{\infty}(2n+1)\tilde{l}_i(\pi_n)\tilde{\pi}_n(\gamma^{-1}(x)\gamma(y)), \quad (20)$$

for $1 \leq i \leq k$. The coefficient functions $\{L_i(x, y)\}_{i=1}^{k}$ comprise the spherical Wiener filter.

4. EXPERIMENTAL RESULTS

The spherical Wiener filter was employed to denoise the 3D Stanford Bunny [9] as shown in Figure 1. The noisy bunny is shown in Figure 1(b). Various projections (height-fields) of the noisy bunny were taken at different locations for the detector placed inside the 3D surface. This is the typical scenario in biomedical imaging for non-invasive surgeries. The samples are regarded as noisy observations on the unit sphere as shown in Figure 1(c) and are Wiener filtered to recover the denoised bunny in Figure 1(d).

Various software packages are available online for computing Spherical Fourier Transform (SFT). The relations in equation (20) are computed for all $n \leq B$, for some positive integer B, also referred to as the *bandwidth* of SFT (see Ch. 9 [5]). The results are shown with $B = 256$. The restored image quality improves with B. However, this improvement is associated with computational penalty. The fast discrete SFT is computed in $O(B^2 \log^2 B)$ operations [10].

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