COMPUTATIONALLY EFFICIENT ESPRIT-LIKE ALGORITHM FOR ESTIMATING QUADRATIC PHASE COUPLING

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ABSTRACT

An eigen-subspace algorithm is proposed for estimating the quadratically-phase-coupled (QPC) frequency pairs in a random harmonic signal corrupted by white noise. A single-parameter matrix-pencil is formed from third order cumulant matrices. The rank reducing numbers of the matrix pencil directly yield the quadratically-phase-coupled frequency pairs. The corresponding generalized eigenvectors are used to determine the coupling strengths. The determination of these rank reducing numbers and generalized eigenvectors is shown to be equivalent to obtaining the eigenvalues and eigenvectors of a single matrix. The algorithm presented here is a considerable improvement over ESPRIT-like algorithm developed in [1]. The original algorithm [1] involved a search overhead of complexity $\mathcal{O}(q^2)$ where q is the number of distinct QPC pairs. The algorithm presented here eliminates any kind of search, which renders it feasible for practical implementations.

***Index Terms*—** QPC, Quadratic phase coupling, ESPRIT-like algorithm, eigen-subspace algorithm.

1. INTRODUCTION

The use of polyspectral analysis in signal processing was motivated by the fact that the second-order statistics are *phase-blind*. The power spectrum of a wide-sense-stationary process gives the distribution of power among the various harmonic components present in it but tells nothing about the phase relations that may exist among these components. On the other hand, higher order spectra retains phase information between harmonic components and hence is particularly useful for applications involving non-Gaussian signals and nonlinear systems. For instance, consider the response of a nonlinear time-invariant system to a superposition of sinusoids (with random statistically independent phases) corrupted by non-gaussian noise. The output is a superposition of sinusoids, the frequencies and phases of which are harmonically related to those of the input. The phase relations among the harmonic components are produced due to nonlinear interactions between the harmonic components of the input signal. This paper presents an algorithm that uses third-order cumulants to estimate phase coupled sinusoids occuring in second-degree nonlinear systems. A second order non-linearity introduces *quadratic phase coupling* which we define now. Consider a stationary harmonic signal

$$x[n] = s[n] + w[n] \tag{1}$$

consisting of sinusoids

$$s[n] = \sum_{i=1}^{p} A_i e^{j(\omega_i n + \phi_i)} \tag{2}$$

in white non-Gaussian noise $w[n]$. A_i and ϕ_i are magnitude and phase, respectively, of the corresponding harmonic component. Define

$$\begin{aligned} \Omega &= \{\omega_1, \ldots, \omega_n\} \\ \Phi &= \{\phi_1, \ldots, \phi_n\}. \end{aligned} \tag{3}$$

Lets denote a harmonic component by the pair (ω_k, ϕ_k). A harmonic component $(\omega_k, \phi_k) \in \Omega \times \Phi$ is said to be ***quadratically phase-coupled*** with $(\omega_l, \phi_l) \in \Omega \times \Phi$ if there exists a third harmonic component $(\omega, \phi) \in \Omega \times \Phi$ in the harmonic process $s[n]$ such that

$$(\omega, \phi) = (\omega_k, \phi_k) + (\omega_l, \phi_l). \tag{4}$$

The problem of identifying quadratic phase coupling finds application in biomedical engineering [4], oceanic engineering [5], acoustics [6] and various other fields. Various non-parametric [3] and parametric [2] methods have been proposed to estimate quadratically-phase-coupled (QPC) pairs. Time-domain approaches were developed in [12]. These methods however suffer from poor resolution and accuracy. Two subspace-based high-resolution methods were developed in [1] which exploit the structure of the third order cumulant of the signal model - the first one involves constructing two "single-parameter" matrix-pencils while second suggests using a single "two-parameter" matrix pencil. Both methods

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however are very intensive computationally. They involve overhead processing of constructing upto q^2 matrices (where q is the number of distinct ordered QPC pairs in the signal), each of size $q \times 2q$, finding their ranks and searching over the two dimensional $\Omega_1 - \Omega_2$ (frequency) space.

The algorithm presented here constructs a ***single "one-parameter"*** matrix pencil to determine both the QPC pairs as well as the coupling strengths via its rank reducing numbers and associated generalized eigenvectors, thereby ***eliminating the overhead computation*** (which is atleast of complexity $\mathcal{O}(q^2)$). The algorithm presented is fast and it retains the high resolution and articulate accuracy features of the original subspace algorithm. This is substantiated by simulations.

2. PROBLEM FORMULATION

The third order cumulants of harmonic signal defined in equation (1) can be shown to have the general form

$$C(\tau,\rho) = C_s(\tau,\rho) + \gamma\delta(\tau)\delta(\rho) \tag{5}$$

where

$$C(\tau,\rho) = \text{Cum}\left\{x^*(n), x(n+\tau), x(n+\rho)\right\}, \tag{6a}$$
$$C_s(\tau,\rho) = \text{Cum}\left\{s^*(n), s(n+\tau), s(n+\rho)\right\}, \tag{6b}$$

are the third order cumulants of $x[n]$ and $s[n]$ (Cum$\{\cdot\}$ is the operator notation for the cumulant [1]) and γ is the skewness of the non-gaussian white noise $w[n]$. From definition of $s[n]$ as a superposition of sinusoids, we get [1]

$$C(\tau,\rho) = \sum_{i=1}^{r}\lambda_i\left(e^{j(\omega_{1,i}\tau+\omega_{2,i}\rho)} + e^{j(\omega_{2,i}\tau+\omega_{1,i}\rho)}\right) + \gamma\delta(\tau)\delta(\rho) \tag{7}$$

where the pairs $\{(\omega_{1,i},\omega_{2,i}) : 1 \le i \le r\}$ are the quadratically phase coupled frequency pairs of the signal and λ_i is the corresponding coupling strength (the magnitude of the two dimensional fourier transform of $C_s(\tau,\rho)$ at the frequency pair $(\omega_{1,i},\omega_{2,i})$ - it equals the product $A_{1,i}A_{2,i}A_{3,i}$). We may assume without loss of generality that $\omega_{1,i} = \omega_{2,i}$ for $1 \le i \le s$ and $\omega_{1,i} \ne \omega_{2,i}$ for $s+1 \le i \le r$. These cumulants can then be expressed in the more convenient form

$$C(\tau,\rho) = \sum_{i=1}^{q}\alpha_i e^{j(\theta_{1,i}\tau+\theta_{2,i}\rho)} + \gamma\delta(\tau)\delta(\rho) \tag{8}$$

where $q = 2r-s$; the coefficients α_i (skewness of the coupled harmonic components [1]) are given by

$$\alpha_i = \begin{cases} 2\lambda_i, & 1 \le i \le s \\ \lambda_i, & s+1 \le i \le r \\ \lambda_{i-r+s}, & r+1 \le i \le 2r-s \end{cases} \tag{9}$$

and the pairs $\{\theta_{1,i},\theta_{2,i}\}_{i=1}^{q}$, given by

$$\begin{array}{lll} \theta_{1,i} = \omega_{1,i} & \theta_{2,i} = \omega_{2,i} & 1 \le i \le r \\ \theta_{1,i} = \omega_{2,i-r+s} & \theta_{2,i} = \omega_{1,i-r+s} & r+1 \le i \le q \end{array} \tag{10}$$

comprise all ordered coupled frequency-pairs of the signal. These ordered pairs are all distinct and the problem is to determine the triplets $(\theta_{1,i},\theta_{2,i},\alpha_i)$ from a finite set of cumulant lags $\{C(\tau,\rho)\}$. Without any loss in generality, we assume $\{\alpha_i\}$ and γ to be real.

3. SUBSPACE BASED ALGORITHM

3.1. Constructing a family of cumulant matrices

Consider the $N^2 \times N^2$ cumulant matrix

$$\mathbf{C}_{a,b} = \sum_{i,j,k,l} C(i-j-a, k-l-b)(\mathbf{u_i}\otimes\mathbf{u_k})(\mathbf{u_j}\otimes\mathbf{u_l})^{\mathbf{T}}$$

where $\otimes$ denotes "Kronecker Product", N is any integer and $\mathbf{u}_i$ is the $N \times 1$ vector having a one in the i^{th} position and zeroes elsewhere. The matrix $\mathbf{C}_{a,b}$ consists of $N \times N$ toeplitz block-matrices. From the structure (8) of the signal cumulants, we find that

$$\mathbf{C}_{a,b} = \sum_{i=1}^{q}\alpha_i e^{j(\theta_{1,i}a+\theta_{2,i}b)}\mathbf{e}(\theta_{1,i})\otimes\mathbf{e}(\theta_{2,i}) + \gamma\mathbf{Z}^a\otimes\mathbf{Z}^b \tag{11}$$

where $\mathbf{e}(\theta) = [1, e^{j\theta}, ..., e^{j(N-1)\theta}]^T$ and $\mathbf{Z}^a$ is an $N \times N$ matrix whose $(i,j)^{th}$ entry is $\delta(i-j-a)$ (i.e. $\mathbf{Z}^a$ has ones on the a^{th} subdiagonal and zeroes elsewhere). The number of distinct coupled frequencies $(\theta_{1,i},\theta_{2,i})$ in the signal is atmost q. Let S denote this set. Assuming that $N \ge q$, it follows (from a basic result about Van-der-Monde vectors) that the vectors $\{\mathbf{e}(\theta) : \theta \in S\}$ are linearly independent. And since the tensor product of two linearly independent sets is linearly independent, the set $\{\mathbf{e}(\theta_1)\otimes\mathbf{e}(\theta_2) : (\theta_1,\theta_2) \in S \times S\}$ is a linearly independent set of vectors in $\mathbb{C}^{N^2}$. Now, the set of coupled frequency pairs is a subset of $S \times S$ and thus, it follows that the vectors

$$\mathbf{a}_i = \mathbf{e}(\theta_{1,i})\otimes\mathbf{e}(\theta_{2,i}) : 1 \le i \le q \tag{12}$$

are linearly independent. Define

$$\mathbf{A} = [\mathbf{a}_1, ..., \mathbf{a}_q], \tag{13a}$$
$$\mathbf{D} = diag[\alpha_1, ..., \alpha_q], \tag{13b}$$
$$\mathbf{\Phi_1} = diag[e^{j\theta_{1,1}}, ..., e^{j\theta_{1,q}}], \tag{13c}$$
$$\mathbf{\Phi_2} = diag[e^{j\theta_{2,1}}, ..., e^{j\theta_{2,q}}]. \tag{13d}$$

With these notations the cumulant matrix (11) can be expressed as

$$\mathbf{C}_{a,b} = \mathbf{AD}(\Phi_1^a\Phi_2^b)^H\mathbf{A}^H + \gamma\mathbf{Z}^a\mathbf{Z}^b. \tag{14}$$

We will form linear combination of the "*cumulant matrix*"

$$\mathbf{C}_x = \mathbf{C}_{0,0} \tag{15}$$

and its "*unit sample delayed versions*"

$$\mathbf{C}_{x,1} = \mathbf{C}_{1,0}, \tag{16a}$$
$$\mathbf{C}_{x,2} = \mathbf{C}_{0,1}. \tag{16b}$$

The corresponding noiseless cumulant matrices are

$$\mathbf{C_s} = \mathbf{ADA^H}, \tag{17a}$$
$$\mathbf{C_{s,1}} = \mathbf{AD\Phi_1^H A^H}, \tag{17b}$$
$$\mathbf{C_{s,2}} = \mathbf{AD\Phi_2^H A^H}. \tag{17c}$$

The following relations now follow from the definitions above

$$\mathbf{C}_x = \mathbf{C}_s + \gamma \mathbf{I}_{M\times M}, \tag{18a}$$
$$\mathbf{C}_{x,1} = \mathbf{C}_{s,1} + \gamma \mathbf{Z} \otimes \mathbf{I}_{N\times N}, \tag{18b}$$
$$\mathbf{C}_{x,2} = \mathbf{C}_{s,2} + \gamma \mathbf{I}_{N\times N} \otimes \mathbf{Z}. \tag{18c}$$

3.2. Signal and Noise Subspaces

The matrix $\mathbf{A}$ has rank q (full column rank) in view of the linear independence of the vectors $\mathbf{a}_i$ defined in (12). Thus, from equation (17a), $\mathbf{C}_s$ is a rank q Hermitian matrix. Denote its nonzero eigenvalues by $\mu_1, ..., \mu_q$ and the corresponding orthonormal eigenvectors by $\mathbf{v}_1, ..., \mathbf{v}_q$. Let $\mathbf{v}_{q+1}, ..., \mathbf{v}_M$, (where $M = N^2$) be an orthonormal basis for the zero eigenvalue subspace of $\mathbf{C}_s$. The eigendecompositions of $\mathbf{C}_s$ and $\mathbf{C}_x$ are respectively given by

$$\mathbf{C}_s = \sum_{i=1}^{q} \mu_i \mathbf{v}_i \mathbf{v}_i^H = \mathbf{E}_s \Lambda_s \mathbf{E}_s^H, \tag{19}$$

$$\begin{aligned} \mathbf{C}_x &= \sum_{i=1}^{q} (\mu_i + \gamma) \mathbf{v}_i \mathbf{v}_i^H + \gamma \sum_{i=q+1}^{M} \mathbf{v}_i \mathbf{v}_i^H \\ &= \mathbf{E}_s (\Lambda_s + \gamma \mathbf{I}_q) \mathbf{E}_s^H + \gamma \mathbf{E}_n \mathbf{E}_n^H, \end{aligned} \tag{20}$$

where

$$\mathbf{E}_s = [\mathbf{v}_1, ..., \mathbf{v}_q], \tag{21a}$$
$$\mathbf{E}_n = [\mathbf{v}_{q+1}, ..., \mathbf{v}_M], \tag{21b}$$
$$\Lambda_s = diag[\mu_1, ..., \mu_q]. \tag{21c}$$

The eigenvalues of $\mathbf{C}_x$ are therefore the diagonal entries of the matrix

$$\Lambda = diag[\mu_1 + \gamma, \ldots, \mu_q + \gamma, \gamma, \ldots, \gamma] \tag{22}$$

and setting $\mathbf{E} = [\mathbf{E}_s | \mathbf{E}_n]$, the eigendecomposition (20) of $\mathbf{C}_x$ can be expressed in the compact notation

$$\mathbf{C}_x = \mathbf{E} \boldsymbol{\Lambda} \mathbf{E}^H. \tag{23}$$

If the noise is Gaussian then $\gamma = 0$. If the noise is weak non-Gaussian, we may assume that $|\alpha_i| >> |\gamma|$ and hence $|\mu_i| >> |\gamma|$. Under this hypothesis (that the signal cumulants are large in comparison to the noise cumulants), γ may be obtained as the least magnitude eigenvalue of $\mathbf{C}_x$ and subtracted off from $\mathbf{C}_x$ according to equation (18a), to yield $\mathbf{C}_s$. We term $\{\mu_i + \gamma : 1 \le i \le q\}$ the signal eigenvalues and γ the noise eigenvalue of $\mathbf{C}_x$. The noise eigenvalue has a multiplicity of $M - q$. The corresponding eigenvectors $\{\mathbf{v}_1, ..., \mathbf{v}_q\}$ which form the columns of $\mathbf{E}_s$ are the signal eigenvectors and $\{\mathbf{v}_{q+1}, ..., \mathbf{v}_M\}$, which are the columns of $\mathbf{E}_n$ are the noise eigenvectors. The subspaces spanned by these vectors, namely $\mathcal{R}(\mathbf{E}_s)$ and $\mathcal{R}(\mathbf{E}_n)$ are the signal and noise subspaces. From (17), the full column rank of $\mathbf{A}$ and non-singularity of $\mathbf{D}$, Φ_1 and Φ_2, it is easy to see that the signal subspace coincides with the subspaces

$$\begin{aligned} \mathcal{R}(\mathbf{C}_s) &= \mathcal{R}(\mathbf{C}_{s,1}) = \mathcal{R}(\mathbf{C}_{s,2}) \\ &= \mathcal{R}(\mathbf{C}_{s,1}^H) = \mathcal{R}(\mathbf{C}_{s,2}^H) = \mathcal{R}(\mathbf{A}), \end{aligned} \tag{24}$$

while the noise subpace is simply

$$\begin{aligned} \mathcal{R}(\mathbf{A})^\perp &= \mathcal{N}(\mathbf{A}^H) = \mathcal{N}(\mathbf{C_s}) \\ &= \mathcal{N}(\mathbf{C_{s,1}}) = \mathcal{N}(\mathbf{C_{s,2}}). \end{aligned} \tag{25}$$

3.3. The Earlier Cumulant Matrix Pencil Algorithm

The identities

$$\begin{aligned} \mathbf{C}_s - \beta \mathbf{C}_{s,1} &= \mathbf{AD}(\mathbf{I} - \beta \Phi_1^H) \mathbf{A}^H, \\ \mathbf{C}_s - \beta \mathbf{C}_{s,2} &= \mathbf{AD}(\mathbf{I} - \beta \Phi_2^H) \mathbf{A}^H, \end{aligned} \tag{26}$$

combined with the fact that matrices $\mathbf{AD}$ and $\mathbf{A}$ have full column ranks imply that for all complex β,

$$\begin{aligned} Rank(\mathbf{C}_s - \beta \mathbf{C}_{s,1}) &= Rank(\mathbf{I} - \beta \Phi_1^H), \\ Rank(\mathbf{C}_s - \beta \mathbf{C}_{s,2}) &= Rank(\mathbf{I} - \beta \Phi_2^H). \end{aligned} \tag{27}$$

This leads us to the following result from [1].

Theorem 3.1. *The rank reducing numbers of the matrix pencils* $\mathbf{C}_s - \beta \mathbf{C}_{s,1}$ *and* $\mathbf{C}_s - \beta \mathbf{C}_{s,2}$ *are* $e^{j\theta_{1,i}}$ *and* $e^{j\theta_{2,i}}$ *respectively, for* $i = 1, ..., q$.

Proof. Refer to [1]. □

So the rank reducing numbers yield the coupled frequencies of the signal. In [1], it was shown that this method of obtaining the rank reducing numbers could be reduced to determining the eigenvalues of a matrix. Pairing of these coupled frequencies was then performed by searching over pairs (β_1, β_2) for which the dimension of

$$W(\beta_1, \beta_2) = \mathcal{N}(\mathbf{C}_s - \beta_1 \mathbf{C}_{s,1}) \cap \mathcal{N}(\mathbf{C}_s - \beta_2 \mathbf{C}_{s,1}) \tag{28}$$

shows an increase. This search is to be performed over at most q^2 pairs. The "new vectors" introduced by this increased dimensionality were used to determine the coupling strengths.

4. THE IMPROVED ALGORITHM

In what follows, we construct a single matrix pencil and show how its rank reducing numbers and generalized eigenvectors can be used to determine the coupled frequency pairs and coupling strengths thus completely avoiding any search.

4.1. Determination of the QPC pairs

The previous algorithm constructed two matrix pencils, one each with a delayed version of the cumulant matrix, and the rank reducing numbers of each pencil were used to estimate the coupled frequencies. However, to determine the coupled pairs, the algorithm had to search over all possible combinations of the q estimated coupled frequencies. The search involved the overhead of constructing the two-parameter matrix pencil (28) and determining its rank for each possible pair. The algorithm presented here avoids this search and related overhead of cimputing cumbersome matrices and estimating their ranks. We form a single one-parameter matrix pencil. The rank reducing numbers of this matrix directly give the *coupled frequency pairs*.

Theorem 4.1. *The rank reducing numbers of the matrix pencil* $\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})$ *are* $e^{j\theta_{1,i}} + e^{j\theta_{2,i}}, 1 \leq i \leq q$. *Each rank reducing number uniquely determines a coupled frequency pair.*

Proof. Using the definitions in equation (17), the matrix pencil can be expressed as

$$\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}) = \mathbf{AD}(\mathbf{I} - \beta(\Phi_1 + \Phi_2)^H)\mathbf{A}^H, \quad (29)$$

Since the matrices $\mathbf{A}$ and $\mathbf{AD}$ have full column ranks, it follows that, for all complex β,

$$Rank\{\mathbf{C_s} - \beta(\mathbf{C_{s,1}} + \mathbf{C_{s,2}})\} = Rank(\mathbf{I} - \beta(\Phi_1 + \Phi_2)^H).$$

The right hand side is less than q if and only if at least one of the diagonal entries of $\mathbf{I} - \beta(\Phi_1 + \Phi_2)$ is zero. This happens if and only if $\beta = (e^{j\theta_{1,i}} + e^{j\theta_{2,i}})^{-1}$ for some i. Or equivalently, if and only if

$$\begin{aligned} \sigma_1 &= cos(\theta_{1,i}) + cos(\theta_{2,i}), \\ \sigma_2 &= sin(\theta_{1,i}) + sin(\theta_{2,i}), \end{aligned} \quad (30)$$

where $\sigma_1 = Real(1/\beta), \sigma_2 = Imag(1/\beta)$. These equations can be inverted to obtain $(\theta_{1,i}, \theta_{2,i})$. Two solution-pairs exist, one of which is obtained by reversing the components of the other. Thus each rank reducing number β uniquely determines a coupled frequency pair. □

We now show that the rank reducing numbers may be obtained as the eigenvalues of a $q \times q$ matrix.

Theorem 4.2. *The rank reducing numbers* β *of* $(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))$ *are precisely the reciprocals of the eigenvalues of the* $q \times q$ *matrix* $\Lambda_s^{-1}\mathbf{E}_s^H(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})\mathbf{E}_s$.

Proof. Since $\mathbf{E}$ is a unitary matrix, the rank of the matrix $\mathbf{E}^H(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))\mathbf{E}$ agrees with the rank of the matrix $\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})$ for all β. However from orthogonality of signal and noise subspaces,

$$\mathbf{A^H E_n} = \mathbf{0}.$$

It follows that

$$\mathbf{C}_s\mathbf{E}_n = \mathbf{C}_{s,1}\mathbf{E}_n = \mathbf{C}_{s,2}\mathbf{E}_n = \mathbf{0}$$

and

$$\mathbf{E}_n^H\mathbf{C}_s = \mathbf{E}_n^H\mathbf{C}_{s,1} = \mathbf{E}_n^H\mathbf{C}_{s,2} = \mathbf{0}.$$

Hence

$$\begin{aligned} &\mathbf{E}^H(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))\mathbf{E} \\ &= \begin{pmatrix} \mathbf{E}_s^H \\ \mathbf{E}_n^H \end{pmatrix} (\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))[\mathbf{E}_s|\mathbf{E}_n] \\ &= \begin{pmatrix} \mathbf{E}_s^H(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))\mathbf{E}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \end{aligned} \quad (31)$$

implying that

$$\begin{aligned} &Rank(\mathbf{E}^H(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))\mathbf{E}) \\ &= Rank(\mathbf{E}_s^H(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))\mathbf{E}_s). \end{aligned} \quad (32)$$

Since $\mathbf{E}_s^H\mathbf{C}_s\mathbf{E}_s = \Lambda_s$, it finally follows that the rank reducing numbers β of $(\mathbf{C}_s - \beta(\mathbf{C}_{s,1} + \mathbf{C}_{s,2}))$ are precisely the reciprocals of the eigenvalues of the $q \times q$ matrix $\Lambda_s^{-1}\mathbf{E}_s^H(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})\mathbf{E}_s$. □

4.2. Determination of the Coupling Strengths

The final result concerns determination of the skewness values. We summarize this:

Theorem 4.3. *Let* β_i *be a rank reducing number of the matrix pencil* $\mathbf{C_s} - \beta(\mathbf{C_{s,1}} + \mathbf{C_{s,2}})$. *Assume that* $(\theta_{1,i}, \theta_{2,i})$ *is the corresponding QPC pair and* $\mathbf{h_i}$ *is the corresponding generalized eigenvector in the signal subspace. If* $\theta_{1,i} = \theta_{2,i}$, *then*

$$\alpha_i = \frac{\mathbf{h_i}^H\mathbf{C_s}\mathbf{h_i}}{|\mathbf{a_i}^H\mathbf{h_i}|^2},$$

where $\mathbf{a_i} = \mathbf{e}(\theta_{1,i}) \otimes \mathbf{e}(\theta_{2,i})$, *while if* $\theta_{1,i} \neq \theta_{2,i}$, *then*

$$\alpha_i = \frac{\mathbf{h_i}^H\mathbf{C_s}\mathbf{h_i}}{|\mathbf{a_i}^H\mathbf{h_i}|^2 + |\mathbf{a_j}^H\mathbf{h_i}|^2},$$

where

$$\begin{aligned}\mathbf{a_i} &= \mathbf{e}(\theta_{1,i}) \otimes \mathbf{e}(\theta_{2,i}), \\ \mathbf{a_j} &= \mathbf{e}(\theta_{2,i}) \otimes \mathbf{e}(\theta_{1,i}).\end{aligned} \tag{33}$$

Proof. Assume β_i is a rank reducing number of the pencil $\mathbf{C_s} - \beta(\mathbf{C_{s,1}} + \mathbf{C_{s,2}})$. Then

$$rank(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}})) < q.$$

Equivalently, $\mathcal{N}(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}}))$ has a dimension more than $M - q$. However by (25),

$$\mathcal{N}(\mathbf{A}^H) \subset \mathcal{N}(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}}))$$

and $\mathcal{N}(\mathbf{A}^H)$ has a dimension $M - q$. So it follows that there exists a nonzero $\mathbf{h_i} \in \mathcal{N}(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}}))$ that also falls in the signal subspace $\mathcal{R}(\mathbf{A})$. Therefore,

$$(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}}))\mathbf{h_i} = \mathbf{0}. \tag{34}$$

Using the definitions in equation (17) and the fact that the matrices $\mathbf{A}$ and $\mathbf{D}$ are have full column rank, we get

$$(\mathbf{I} - \beta_i(\Phi_1^H + \Phi_2^H))\mathbf{A}^H\mathbf{h_i} = \mathbf{0}. \tag{35}$$

Let $(\theta_{1,i}, \theta_{2,i})$ be the QPC pair corresponding to the rank reducing number β_i. If $\theta_{1,i} = \theta_{2,i}$, then this QPC pair occurs just once, as the i^{th} entry, among the corresponding diagonal entries of Φ_1 and Φ_2. In this case all the diagonal entries of $\mathbf{I} - \beta_i(\Phi_1^H + \Phi_2^H)$ except the i^{th} one are non-zero and we get from (35),

$$\mathbf{a_k}^H\mathbf{h_i} = 0, \quad k \neq i. \tag{36}$$

Since $\mathbf{h_i}$ is a nonzero vector in the signal subspace, it follows that $\mathbf{A}^H\mathbf{h_i}$ is nonzero and hence in view of (36), $\mathbf{a_i}^H\mathbf{h_i} \neq 0$. Thus

$$\mathbf{h_i}^H\mathbf{C_s}\mathbf{h_i} = |\mathbf{a_i}^H\mathbf{h_i}|^2\alpha_i,$$

from which the skewness value α_i may be determined. Suppose $\theta_{1,i} \neq \theta_{2,i}$. Then, this QPC pair occurs twice as corresponding diagonal entries of Φ_1 and Φ_2, in different orders, i.e., once as the ordered pair $(\theta_{1,i}, \theta_{2,i})$ and as $(\theta_{2,i}, \theta_{1,i})$. The first is the i^{th} diagonal entry, let the second one be the j^{th}. In this case, all the diagonal entries of $\mathbf{I} - \beta_i(\Phi_1^H + \Phi_2^H)$ except the i^{th} and j^{th} ones are nonzero and it follows from (35) that

$$\mathbf{a_k}^H\mathbf{h_i} = 0, \quad k \neq i, j.$$

Since $\mathbf{A}^H\mathbf{h_i}$ is nonzero, it must therefore follow that at least one of the complex numbers $\mathbf{a_i}^H\mathbf{h_i}$ and $\mathbf{a_j}^H\mathbf{h_i}$ must be nonzero. Since the i^{th} and j^{th} diagonal entries corresponding to the same QPC pair, $\alpha_i = \alpha_j$ and thus,

$$\mathbf{h_i}^H\mathbf{C_s}\mathbf{h_i} = \alpha_i(|\mathbf{a_i}\mathbf{h_i}|^2 + |\mathbf{a_j}\mathbf{h_i}|^2),$$

from which the skewness can be obtained. □

The problem of determining an appropriate $\mathbf{h_i}$ corresponding to each rank reducing number may be reduced to that of determining an eigenvector just as the problem of determining the rank reducing number β_i was reduced to that of determining an eigenvalue.

Theorem 4.4. *Let $\mathbf{x_i}$ be an eigenvector of $\Lambda_s^{-1}\mathbf{E}_s^H(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})\mathbf{E}_s$ with eigenvalue β_i and $(\theta_{1,i}, \theta_{2,i})$ as the associated QPC pair. The vector $\mathbf{h_i}$ of Theorem 4.3 is then given by $\mathbf{E_s}\mathbf{x_i}$.*

Proof. We have seen that the reciprocal of the rank reducing number, $1/\beta_i$, is an eigenvalue of $\Lambda_s^{-1}\mathbf{E}_s^H(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})\mathbf{E}_s$. Let $\mathbf{x_i}$ be a corresponding eigenvector. Thus,

$$(\mathbf{\Lambda_s} - \beta_i\mathbf{E_s}^H(\mathbf{C}_{s,1} + \mathbf{C}_{s,2})\mathbf{E_s})\mathbf{x_i} = \mathbf{0}.$$

This may be expressed as

$$\mathbf{E_s}^H(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}}))\mathbf{E_s}\mathbf{x_i} = \mathbf{0},$$

Using the nonsingularity of $[\mathbf{E_s}|\mathbf{E_n}]$ and the fact

$$\mathbf{E_n}^H(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}})) = \mathbf{0},$$

we deduce that

$$(\mathbf{C_s} - \beta_i(\mathbf{C_{s,1}} + \mathbf{C_{s,2}}))\mathbf{E_s}\mathbf{x_i} = \mathbf{0}.$$

The sought after vector $\mathbf{h_i}$ in the signal subspace is therefore given by $\mathbf{h_i} = \mathbf{E_s}\mathbf{x_i}$. □

5. SIMULATION RESULTS

The simulation results for the computationally efficient ESPRIT (CE-ESPRIT) like algorithm are presented in this section. The harmonic process consists of eight components with 2 coupled pairs and 2 uncoupled sinusoids. Coupled frequencies are:

$$\begin{aligned}(\theta_{1,1}, \theta_{2,1}) &= (1.5, 2.5) \quad \theta_{3,1} = 4.0 \\ (\theta_{1,2}, \theta_{2,2}) &= (1.0, 3.5) \quad \theta_{3,2} = 4.5\end{aligned}$$

and the uncoupled frequencies are:

$$\theta_{4,1} = 2.1991, \theta_{4,2} = 2.8274.$$

The magnitude of coupled triplets is $A = 3$, while magnitude of rest of the frequencies is 1 (the coupling strength for each QPC pair is then A^3). The third order cumulants are estimated using the direct method [3]. The variance of non gaussian noise is 0.03 and the skewness is 0.014. N is chosen to be 5. The estimation results are tabulated above for five different realizations of the noisy harmonic process. The estimates of the coupled frequencies as well as mean and variance of the CE-ESPRIT estimator are tabulated in Table1 for the first pair and in Table2 for the second pair. Note that the QPC frequency estimates and estimates for the coupling strengths are unbiased and the variance is remarkably low.

No.	$\theta_{1,1}$	$\theta_{2,1}$	Skewness (α_1)
1	1.5158	2.4991	26.4783
2	1.5432	2.4931	25.9927
3	1.4619	2.5155	27.3349
4	1.4741	2.5065	28.7640
5	1.5257	2.4934	26.5505
μ	1.5041	2.5015	27.0241
σ^2	0.0010	0.0001	0.9420

Table 1. Estimation results for the first QPC pair.

No.	$\theta_{1,2}$	$\theta_{2,2}$	Skewness (α_2)
1	1.0036	3.5048	27.9066
2	1.0204	3.5260	26.9234
3	1.0148	3.5040	29.8577
4	1.0031	3.5017	26.7313
5	1.0084	3.5100	27.0743
μ	1.0101	3.5093	27.6987
σ^2	0.00004	0.0001	1.3263

Table 2. Estimation results for the second QPC pair.

6. CONCLUDING REMARKS

A computationally efficient ESPRIT-like algorithm is presented that uses third order cumulants to identify quadratically phase coupled pairs in a harmonic process corrupted by non-gaussian noise. The algorithm presented here is a considerable improvement over the original algorithm proposed in [1] since it avoids any kind of search, not only over the whole $\Omega_1 - \Omega_2$ plane but also over the q pairs of frequency. In addition the construction of cumbersome matrices and rank calculation of these matrices (as in [1]) for search purpose is avoided. This makes the algorithm extremely fast and feasible for practical implementation. The algorithm also retains the high resolution features of the original algorithm [1] and the immunity to noise is also maintained.

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