WIENER FILTER FOR ISOTROPIC SIGNAL FIELDS

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ABSTRACT

The images of a three dimensional scene taken at varying depths are regarded as reference stochastic processes on the unit sphere to formulate the normal equations for estimating the image at any given depth. These comprise a set of coupled linear integral equations on the unit sphere for the filter bank. The coefficient functions of these equations are the scene correlations which are assumed to be known. By exploiting properties of the action of the rotation group on the unit sphere, this set of coupled integral equations is converted into a set of Wiener-Hopf convolutional integral equations on the rotation group. These are solved using the Peter-Weyl theory of the Fourier transform on the rotation group.

1. INTRODUCTION

Recently, there has been a lot of interest in signal processing for signals defined on the sphere. A major motivation comes from recent advances in the omni-directional imaging techniques that capture the three dimensional scene on the sphere. Such systems are typically realized by catadioptric systems that incorporate convex reflectors with the camera [1, 2]. The three dimensional scene may also be captured using a pair of fisheye lenses [3]. A real time omnidirectional camera incorporating catadioptric module as well as fisheye module thereby providing a full spherical field of view was presented in [4]. These vision systems find applications in robotics [5], video surveillance, medical imaging and automatic face recognition.

In image processing and computer vision, researchers have looked at spherical diffusion techniques for smoothing and denoising 3-D surface data available through modern data acquisition [6]. The 3-D surface data can be considered as height-field on the sphere for *star-shaped* 3-D objects [6]. Alternatively, the 3D surface data may be transformed to spherical signals by clever parametrization [7]. The filters used for spherical diffusion are typically smooth gaussian kernels with low pass characteristics. We develop a wiener filter on the unit sphere that is optimal in the minimum mean squared error sense. The objective is to present optimal reconstruction and denoising techniques for the omni-directional images and 3-D surface data that can be represented as functions or images on the sphere.

This work is also motivated by recent biological findings that human visual system incorporates rotation and pose invariance in recognition. It is remarkable that even in severely degraded conditions the human visual system performs recognition with proficiency [8]. There have been a few studies to investigate the neural response to blurred and rotated images. The findings in [9] suggest that a subset of neurons in dorsolateral prefrontal cortex are rotation invariant. And in somewhat related work, the localization and navigation tasks were suggested to be based on matched filters implemented by neurons that have spherical receptive fields [10].

The spherical wiener filtering problem is formulated in section 2. In section 3 the Wiener-Hopf equations are derived on a homogeneous space. The theory is developed in complete generality for functions defined on a manifold M under the transitive action of a compact group G. The Wiener-Hopf equations on the unit sphere comprise a set of coupled linear integral equations which are expressed as convolutional integral equations on the rotation group. The integral equations are expressed in Fourier domain as discrete sums over the irreducible representations of the rotation group using the Peter-Weyl theory [11]. The analysis is then specialized in section 4 to the case where the manifold is the sphere S^2 and the group is the three dimensional rotation group [12].

2. PROBLEM FORMULATION

Assume that the camera or the imaging device is located at the origin of the three dimensional coordinate system. The object function of a changing scene in 3-D is given by a mapping $\xi_0 : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. For each $\mathbf{x} \in \mathbb{R}^3$, $t \in \mathbb{R}$, $\xi_0(\mathbf{x}, t)$ represents the intensity of the light at the given location $\mathbf{x}$ at time t. In polar coordinates, the position of a point $\mathbf{x}$ in the object is given by the triplet (d, θ, ϕ) where d represents the depth of the point, i.e. the radial distance of the point in question from the detector (or camera) and $\theta \in [0, \pi]$ represents the colatitude and $\phi \in [0, 2\pi)$ is the longitude. With respect to the standard basis for $\mathbb{R}^3$, the rotation about the origin can be represented by an orthogonal 3×3 real-valued matrix with determinant unity [12]. In other words, the rotation group can be represented by the special orthogonal group $SO(3)$.

The scene captured by the camera is possibly corrupted by noise. It is assumed in this paper that the noise is *isotropic*, i.e.

the noise correlation at any two points on a sphere depends only on the rotation connecting the two points. Note that for signals defined on the real line the notion of isotropic signal fields is same as that of wide sense stationarity. For statistical characterization of isotropic noise on the sphere, consider the fluctuation signal field

$$\xi(d,\theta,\phi,t) = \xi_0(d,\theta,\phi,t) - \mathbb{E}\left[\xi_0(d,\theta,\phi,t)\right]$$

at a radial distance d, collected over the whole of the unit sphere $(\theta,\phi) \in S^2$. Assume that for any two depths d_1 and d_2, the corresponding fluctuation signal fields $\xi(d_1,\cdot)$ and $\xi(d_2,\cdot)$ are jointly wide sense $SO(3)\times\mathbb{R}$ stationary in the sense that the correlations

$$R_{\xi\xi}(d_1,\theta_1,\phi_1,t_1;d_2,\theta_2,\phi_2,t_2) = \mathbb{E}\left[\xi(d_1,\theta_1,\phi_1,t_1) \cdot \xi(d_2,\theta_2,\phi_2,t_2)\right] \quad (1)$$

satisfy the invariance condition

$$R_{\xi\xi}(d_1,\theta_1,\phi_1,t_1;d_2,\theta_2,\phi_2,t_2) = R_{\xi\xi}(d_1,R(\theta_1,\phi_1),t_1+\tau; d_2,R(\theta_2,\phi_2),t_2+\tau)$$

for all rotations $R \in SO(3)$ acting on S^2 and time-translations $\tau \in \mathbb{R}$.

By Cramer's spectral representation theorem [13], at a fixed point in the space, the time varying signal ξ can be expressed as

$$\xi(d,\theta,\phi,t) = \int_{-\infty}^{\infty} \hat{\xi}(d,\theta,\phi,\nu)e^{j2\pi\nu t}d\nu \quad (2)$$

where $\hat{\xi}$ denotes the Fourier transform of ξ and satisfies the relationship

$$R_{\xi\xi}(d_1,\theta_1,\phi_1,t+\tau;d_2,\theta_2,\phi_2,t) = \iint \mathbb{E}[\hat{\xi}(d_1,\theta_1,\phi_1,\mu)\overline{\hat{\xi}(d_2,\theta_2,\phi_2,\nu)}]e^{j2\pi(\mu(t+\tau)-\nu t)}d\mu d\nu.$$

It thus follows that

$$\mathbb{E}[\hat{\xi}(d_1,\theta_1,\phi_1,\mu)\overline{\hat{\xi}(d_2,\theta_2,\phi_2,\nu)}] = S_{\xi\xi}((d_1,\theta_1,\phi_1),(d_2,\theta_2,\phi_2),\mu)\cdot\delta(\mu-\nu)$$

where $S_{\xi\xi}$ denotes the power spectral density of ξ,

$$S_{\xi\xi}((d_1,\theta_1,\phi_1),(d_2,\theta_2,\phi_2),\nu) = \int_{-\infty}^{\infty} R_{\xi\xi}(d_1,\theta_1,\phi_1,\tau;d_2,\theta_2,\phi_2,0)e^{-j2\pi\nu\tau}d\tau.$$

The equation above is the Wiener-Khinchin relationship for the $SO(3)\times\mathbb{R}$-stationary signals. This characterization is important since it leads to the following elementary but useful result.

Theorem 1. *For every $R \in SO(3)$*

$$S_{\xi\xi}((d_1,R(\theta_1,\phi_1)),(d_2,R(\theta_2,\phi_2)),\nu) = S_{\xi\xi}((d_1,\theta_1,\phi_1),(d_2,\theta_2,\phi_2),\nu). \quad (3)$$

The theorem states that for fixed ν, the process $\hat{\xi}((d,\theta,\phi),\nu)$ has the joint $SO(3)$ wide sense stationary property as d varies whereas for different frequencies μ,ν the corresponding processes $\hat{\xi}((d,\theta,\phi),\mu)$ and $\hat{\xi}((d,\theta,\phi),\nu)$ are uncorrelated,

$$\mathbb{E}[\hat{\xi}((d_1,\theta_1,\phi_1,\mu))\overline{\hat{\xi}((d_2,\theta_2,\phi_2),\nu)}] = 0$$

for $\mu \neq \nu$. Consequently, we can do processing in the frequency domain and restrict the attention to a fixed frequency ν_0 thereby ignoring signal at other frequencies when reconstructing the signal. This observation can be stated more clearly by assuming a hypothetical ideal bandpass optical filter.

Assume that a camera is focused to take pictures at depth d and orientation (θ,ϕ), the point under focus being given in polar coordinates by (d,θ,ϕ). Suppose that a bandpass filter is placed before the lens so as to receive the signal only in the frequency range $[\nu_0-\Delta\nu,\nu_0+\Delta\nu]$. The frequency response of the filter is denoted by $H(\nu)=1$ for $|\nu-\nu_0|\leq\Delta\nu$ and $H(\nu)=0$ otherwise. The impulse response of the associated filter is $h(t) = e^{j2\pi\nu_0 t}\frac{\sin(2\pi\Delta\nu t)}{\pi t}$. The *received signal* is given by

$$\begin{aligned}\eta(d,\theta,\phi,t) &= \int_{-\infty}^{\infty}\xi(d,\theta,\phi,t-\tau)h(\tau)d\tau \\ &= \int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}\hat{\xi}((d,\theta,\phi),\nu)e^{j2\pi\nu t}d\nu \\ &\approx e^{j2\pi\nu_0 t}\int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}\hat{\xi}((d,\theta,\phi),\nu)d\nu\end{aligned}$$

assuming $\Delta\nu \ll \nu_0$. The quantity of interest here is the complex envelope of the received signal given by

$$\tilde{\eta}((d,\theta,\phi),\nu_0) = \int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}\hat{\xi}((d,\theta,\phi),\nu)d\nu,$$

which will also be referred to as the *recorded signal*. The correlation function of the process $\{\tilde{\eta}((d,\theta,\phi),\nu_0)\}$ is

$$\begin{aligned}&\mathbb{E}[\tilde{\eta}((d_1,\theta_1,\phi_1),\nu_0)\,\overline{\tilde{\eta}((d_2,\theta_2,\phi_2),\nu_0)}\,] \\ &= \int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}\int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}\mathbb{E}\left[\hat{\xi}((d_1,\theta_1,\phi_1),\nu)\cdot\overline{\hat{\xi}((d_2,\theta_2,\phi_2),\mu)}\right]d\nu d\mu \\ &= \int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}\int_{\nu_0-\Delta\nu}^{\nu_0-\Delta\nu}S_{\xi\xi}((d_1,\theta_1,\phi_1),(d_2,\theta_2,\phi_2),\nu)\cdot\delta(\nu-\mu)\,d\nu d\mu \\ &\approx 2\Delta\nu\cdot S_{\xi\xi}((d_1,\theta_1,\phi_1),(d_2,\theta_2,\phi_2),\nu_0).\end{aligned}$$

Thus, the recorded signal field $\{\tilde{\eta}((d,\theta,\phi),\nu_0)\}$ is jointly wide sense stationary in the parameter d with respect to $SO(3)$ action on S^2,

$$\begin{aligned}&\mathbb{E}\left[\tilde{\eta}((d_1,\theta_1,\phi_1),\nu_0)\,\overline{\tilde{\eta}((d_2,\theta_2,\phi_2),\nu_0)}\right] \\ &= \mathbb{E}\left[\tilde{\eta}((d_1,R(\theta_1,\phi_1)),\nu_0)\,\overline{\tilde{\eta}((d_2,R(\theta_2,\phi_2)),\nu_0)}\right],\end{aligned}$$

for every $R \in SO(3)$.

When the camera is focussed to take pictures in the frequency band $[\nu_0 - \Delta\nu, \nu_0 + \Delta\nu]$ by an appropriate narrow bandpass filter at depths $d_1, \ldots, d_k$, the recorded image signals are $\{\tilde{\eta}((d_j, \theta, \phi), \nu_0) | (\theta, \phi) \in S^2, j = 1, \ldots, k\}$. These recorded signals can be viewed as k stochastic processes whose random variables are indexed by points (θ, ϕ) on the surface of the unit sphere. The aim is to construct a linear estimate of the signal field $\{\tilde{\eta}((d, \theta, \phi), \nu_0) | (\theta, \phi) \in S^2\}$ which is the complex envelope of the received signal at depth d within the frequency band $[\nu_0 - \Delta\nu, \nu_0 + \Delta\nu]$.

It is a consequence of the remark after Theorem 1 that the recorded signals at two distinct frequencies ν_0 and ν_0' are uncorrelated and hence it is adequate to restrict the attention to a fixed frequency ν_0. In other words, in order to estimate $\tilde{\eta}(d, \theta, \phi, \nu_0)$ at depth d, the recorded signals at frequencies $\nu \neq \nu_0$ can be ignored.

Inspired by the group theoretic methods in image processing as outlined in [14] this problem is solved by viewing the rotation group $SO(3)$ as a Lie group acting on the manifold S^2, the unit sphere in $\mathbb{R}^3$, and exploiting the harmonic analysis based on the irreducible representations of G. In order to achieve this, a general theory of Wiener filter is first developed for such group actions on manifolds and an explicit formula for the Wiener filter in the case of $SO(3)$ action on S^2 is obtained as a special application.

3. WIENER HOPF EQUATIONS ON HOMOGENEOUS SPACE

Let G be a compact Lie group acting transitively on a manifold M. Suppose that $\{\eta_1(x), \ldots, \eta_k(x), \xi(x) | x \in M\}$ is a collection of $k+1$ real or complex-valued random processes. The variables η_j denote the k observations or samples on the manifold and ξ denotes the un-observed signal that needs to be reconstructed based on the samples. The random processes are assumed to be zero mean,

$$\mathbb{E}[\xi(x)] = 0, \;\; \mathbb{E}[\eta_i(x)] = 0 \tag{4}$$

and jointly wide sense G-stationary, i.e. the correlation functions satisfy

$$\begin{aligned} \mathbb{E}[\xi(x)\overline{\xi(y)}] &= R(x,y) = R(gx,gy) \\ \mathbb{E}[\xi(x)\overline{\eta_j(y)}] &= R_j(x,y) = R_j(gx,gy) \\ \mathbb{E}[\eta_i(x)\overline{\eta_j(y)}] &= R_{ij}(x,y) = R_{ij}(gx,gy) \end{aligned} \tag{5}$$

for all $g \in G$, $x, y \in M$ and $i, j = 1, \ldots, k$. The best linear estimate $\hat{\xi}(x)$ of the input process $\xi(x)$ based on η_j's is

$$\hat{\xi}(x) = \sum_{j=1}^{k} \int_M L_j(x,y)\eta_j(y)dy \tag{6}$$

where integration is with respect to the unique G-invariant probability measure on the compact manifold M and $L_j(x,y)$ are unknown functions to be determined. To ensure that $\hat{\xi}(x)$ is G-stationary process, $L_j(x,y)$ should be G-invariant, i.e. $L_j(gx, gy) = L_j(x,y)$ for $g \in G, \; x, y \in M$.

By the orthogonality principle, the mean square error (MSE) $\mathbb{E}|\xi(x) - \hat{\xi}(x)|^2$ is minimized when

$$\mathbb{E}[(\xi(x) - \hat{\xi}(x))\, \overline{\eta_j(y)}\,] = 0 \tag{7}$$

for $j = 1, \ldots, k$ and $x, y \in M$. Using equation (6) and definitions in (5), this gives the following normal equations

$$R_j(x,y) = \sum_{i=1}^{k} \int_M L_i(x,z) R_{ij}(z,y) dz \tag{8}$$

for $1 \leq j \leq k$ and $x, y \in M$.

We need to find G-invariant functions that satisfy these coupled linear integral equations on the manifold. These equations can be expressed as relations on the group G as follows.

Fix an 'origin' x_0 in M. Then, by the transitivity of the group action, there exist $g_1, g_2 \in G$ such that $x = g_1 x_0$ and $y = g_2 x_0$. Then (8) can be expressed as

$$R_j(g_1x_0, g_2x_0) = \sum_{i=1}^{k} \int_M L_i(g_1x_0, z) R_{ij}(z, g_2x_0) dz. \tag{9}$$

Using the fact that the map $g \to gx_0$ from G into M takes the normalised Haar measure of G to the unique G-invariant probability measure on M, the integral in equation above can be transformed into an integral on G, giving

$$R_j(g_1x_0, g_2x_0) = \sum_{i=1}^{k} \int_G L_i(g_1x_0, gx_0) R_{ij}(gx_0, g_2x_0) dg$$

for $g_1, g_2 \in G$ where dg indicates integration with respect to the Haar measure on G. It is convenient to use different notation when viewing the correlation functions as functions on the group. Lets define $\phi_i(g) = R_i(x_0, gx_0)$, $\psi_{ij}(g) = R_{ij}(x_0, gx_0)$ and $l_i(g) = L_i(x_0, gx_0)$. Then using the invariance conditions its easily verified that

$$\phi_j(g) = \phi_j(g_1^{-1}g_2) = \sum_{i=1}^{k} \int_G l_i(g')\psi_{ij}(g'^{-1}g) dg', \;\; g \in G \tag{10}$$

which can be expressed as a matrix convolution

$$\phi^T(g) = \int_G l^T(g')\psi(g'^{-1}g) dg', \;\; g \in G \tag{11}$$

in a compact notation where $\phi = [\phi_1, \ldots, \phi_k]$, $l = [l_1, \ldots, l_k]^T$ and $\psi = [\psi_{ij}]_{k \times k}$. Note that these convolutional integral equations are the Wiener-Hopf equations (8), expressed in terms of integrals on the group G. The convolutional integral equations in (10) will be solved using the Peter-Weyl theory on the compact group G [14].

Let $\hat{G}$ denote the set of all inequivalent irreducible unitary representations of G. Then by the Peter-Weyl theorem, $\{\sqrt{d_\pi}\pi_{\alpha,\beta}(g)|1 \leq \alpha,\beta \leq d_\pi, \pi \in \hat{G}\}$ is a complete orthonormal basis for $L^2(G)$, the space of all square integrable functions on G, where d_π denotes the dimension of the irreducible representation π. Then for any $f \in L^2(G)$,

$$f(g) = \sum_{\pi,\alpha,\beta} d_\pi \langle f, \pi_{\alpha\beta}\rangle \pi_{\alpha\beta}(g) \tag{12}$$

where

$$\langle f, \pi_{\alpha\beta}\rangle = \int_G f(g)\overline{\pi_{\alpha\beta}(g)}dg. \tag{13}$$

In compact notation the $d_\pi \times d_\pi$ matrix $\hat{f}(\pi) = \int_G f(g)\pi^*(g)dg$ is defined to be the *Fourier transform* of f at frequency $\pi \in \hat{G}$. Similarly, equation (12) can be expressed compactly as

$$f(g) = \sum_{\pi\in\hat{G}} d_\pi \mathrm{Tr}\left(\hat{f}(\pi)\pi(g)\right) \tag{14}$$

which defines the *inverse Fourier transform* of $\hat{f}$ at g.

Taking the Fourier transform on both sides of (10) yields

$$\begin{aligned}
\hat{\phi}_j(\pi) &= \int_G \phi_j(g)\pi^*(g)\, dg \\
&= \sum_{i=1}^{k} \int_{G\times G} l_i(g')\psi_{ij}(g'^{-1}g)\pi^*(g)\, dg'dg \\
&= \sum_{i=1}^{k} \int_{G\times G} l_i(g')\psi_{ij}(h)\pi^*(g'h)\, dg'dh \\
&= \sum_{i=1}^{k} \int_{G\times G} l_i(g')\psi_{ij}(h)\pi^*(h)\pi^*(g')\, dhdg' \\
&= \sum_{i=1}^{k} \hat{l}_i(\pi)\, \hat{\psi}_{ij}(\pi)
\end{aligned} \tag{15}$$

for $j = 1,\ldots,k$. This gives a system of linear equations which can be solved for the unknown $\hat{l}_i(\pi)$. The filter responses $L_i(x,y)$ can then be obtained as follows. Since G is transitive on M, construct a measurable cross-section $\gamma : M \to G$ such that $\gamma(x)x_0 = x \;\forall\; x \in M$, where $\gamma(x)x_0$ denotes the action of $\gamma(x)$ on x_0. Then

$$\begin{aligned}
L_i(x,y) &= L_i(\gamma(x)x_0, \gamma(y)x_0) \\
&= L_i(x_0, \gamma^{-1}(x)\gamma(y)x_0) \\
&= l_i(\gamma^{-1}(x)\gamma(y))
\end{aligned} \tag{16}$$

for $1 \leq i \leq k$ and $x, y \in M$, where l_i is the inverse Fourier transform of $\hat{l}_i$. The functions $\{L_i(x,y)\}_{i=1}^k$ comprise the Wiener filter and can be used in equation (6) to obtain the estimates $\hat{\xi}$.

4. SPHERICAL WIENER FILTER

The wiener filter is now specialized to the setting in section 2 where the manifold M is the unit sphere in $\mathbb{R}^3$,

$$S^2 = \left\{x = (x_1, x_2, x_3)^T : x_1^2 + x_2^2 + x_3^2 = 1,\; x_i \in \mathbb{R}\right\}$$

with the action of the rotation group $SO(3)$.

4.1. Parameterizing through $SU(2)$

The representations of $SO(3)$ are studied as a subset of the representations of the special unitary group, $SU(2)$, consisting of all 2×2 complex unitary matrices of determinant unity,

$$SU(2) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\right\}.$$

The group $SU(2)$ is the double cover of $SO(3)$ (there is a two-to-one surjective homomorphism from $SU(2)$ to $SO(3)$) and it acts on S^2 as follows. For $\mathbf{x} \in \mathbb{R}^3$ define the Hermitian matrix

$$H(\mathbf{x}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, \;\; \mathbf{x}^T = (x_1, x_2, x_3).$$

H is a linear isomorphism from $\mathbb{R}^3$ onto the real linear space of 2×2 Hermitian matrices of zero trace and $\det H(x) = -(x_1^2 + x_2^2 + x_3^2)$. For any $g \in SU(2)$, $gH(x)g^*$ is again a Hermitian matrix of trace zero and hence has the form $H(x')$ for some $x' \in \mathbb{R}^3$. The map $\mathbf{x} \to \mathbf{x}'$ is thus linear. Also $\det H(x') = (\det g)\, detH(x)(\det g^*) = detH(x)$, therefore $x_1^2 + x_2^2 + x_3^2 = x_1'^2 + x_2'^2 + x_3'^2$ and the map $\mathbf{x} \to \mathbf{x}'$ is a linear isometry. The connectedness of $SU(2)$ implies that this isometry is a proper rotation $R(g)$ which, in particular, leaves S^2 invariant, thus $R(g) \in SO(3)$. For

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix},$$

elementary algebra gives

$$R(g) = \begin{pmatrix} Re(a^2 - b^2) & Im(a^2 + b^2) & -2Re(ab) \\ Im(b^2 - a^2) & Re(a^2 + b^2) & 2Im(ab) \\ 2Re(\bar{a}b) & 2Im(a\bar{b}) & |a|^2 - |b|^2 \end{pmatrix}.$$

Furthermore, $R(g_1g_2) = R(g_1)R(g_2)$ for all $g_1, g_2 \in SU(2)$. Define the group action of $SU(2)$ on S^2 as $(g, \mathbf{x}) \to g \cdot \mathbf{x} = R(g)\mathbf{x}$.

4.2. Measurable cross-section $\gamma : M \to G$

Choose the north pole $\mathbf{x}_0 = (0,0,1) \in S^2$ as its origin and define the map $\gamma : S^2 \to SU(2)$ by

$$\gamma(\mathbf{x}) = \begin{pmatrix} \alpha(\mathbf{x}) & \beta(\mathbf{x}) \\ -\bar{\beta}(\mathbf{x}) & \bar{\alpha}(\mathbf{x}) \end{pmatrix} \tag{17}$$

where

$$\alpha(\mathbf{x}) = \left(\frac{1+x_3}{2}\right)^{\frac{1}{2}}, \ \beta(\mathbf{x}) = -\frac{x_1 - ix_2}{[2(1+x_3)]^{\frac{1}{2}}}. \tag{18}$$

Then $\gamma(\mathbf{x}) \cdot \mathbf{x}_0 = R(\gamma(\mathbf{x}))\mathbf{x}_0 = \mathbf{x}$. In particular, $SU(2)$ acts transitively on S^2.

4.3. Irreducible representations of $SU(2)$

The irreducible representations of $SU(2)$ are indexed by half integers $j \in \{0, 1/2, 1, 3/2, \ldots\}$. The representation space of π_j is $(2j+1)$-dimensional vector space of all homogeneous polynomials $p(z_1, z_2)$ of degree $2j$ in two variables. Following the development in [15], for $g = \gamma(\mathbf{x})$, as given in (17), the irreducible representation, π_j, is given as

$$\pi_j(g) = \pi_j(\gamma(\mathbf{x})) = \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r}^2 |\alpha(\mathbf{x})|^{2r} |\beta(\mathbf{x})|^{2(j-r)}.$$

4.4. Wiener filter on the sphere

Define functions ρ_j on the unit sphere S^2 as

$$\rho_j(\mathbf{x}) = \pi_j(\gamma(\mathbf{x})). \tag{19}$$

Then the Fourier transform for $\phi_p(g) = R_p(\mathbf{x}_0, g\mathbf{x}_0)$ at the frequency $\pi = \pi_j$ can be written as

$$\begin{aligned} \hat{\phi}_p(\pi_j) &= \int_{SU(2)} R_p(\mathbf{x}_0, g\mathbf{x}_0)\overline{\pi_j(g)}dg \\ &= \int_{S^2} R_p(\mathbf{x}_0, \mathbf{x})\rho_j(\mathbf{x})d\sigma(\mathbf{x}) \end{aligned} \tag{20}$$

where $d\sigma$ denotes integration with respect to the normalized area measure on S^2,

$$d\sigma(\mathbf{x}) = \frac{dx_1 dx_2}{|x_3|}, \ \ x_1^2 + x_2^2 + x_3^2 = 1. \tag{21}$$

Similarly,

$$\hat{\psi}_{pq}(\pi_j) = \int_{S^2} R_{pq}(\mathbf{x}_0, \mathbf{x})\rho_j(\mathbf{x})d\sigma(\mathbf{x}). \tag{22}$$

Note that R_p and R_{pq} are the correlation functions. Thus the normal equations

$$\hat{\phi}_q(\pi_j) = \sum_{p=1}^{k} \hat{l}_p(\pi_j)\hat{\psi}_{pq}(\pi_j) \tag{23}$$

can be solved for $\hat{l}_p(\pi_j)$ and the inverse Fourier transform of $\hat{l}_p(\pi_j)$ gives

$$L_p(x, y) = \sum_{j=0}^{\infty} (2j+1)\hat{l}_p(\pi_j)\pi_j(\gamma^{-1}(x)\gamma(y)) \tag{24}$$

for $1 \leq p \leq k$. The functions $L_p(x, y)$ yield the minimum mean-square-error estimate of ξ based on η's as outlined in equation (6). The implementation of wiener filter is not discussed here due to space constraints.

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