INDIAN INSTITUTE OF TECHNOLOGY DELHI

Stationarity condition for Fractional sampling filters

by

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Report submitted in fulfillment of the requirement of the degree of Masters of Technology

under guidance of

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Certificate

This is to certify that the report titled **"Stationarity condition for Fractional sampling filters"** being submitted by Pushpendre Rastogi to the Department of Electrical Engineering, Indian Institute of Technology, Delhi, for the award of the degree of Masters of Technology, is a record of bona-fide research work carried out by him under my guidance and supervision. In my opinion, the report has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this report have not been submitted to any other university or institute for the award of any degree or diploma.

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Declaration of Authorship

I, Pushpendre Rastogi, declare that this report titled, 'Stationarity condition for Fractional sampling filters' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

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Abstract

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Cyclostationary processes are non-stationary processes whose statistical characteristics repeat in time with a period greater than 1. In general Cyclostationary processes when passed through LTI systems remain Cyclostationary with the same period. However there are special cyclostationary signals which when passed through specific LTI filters can become stationary. One such type of cyclostationary signal is the output of an interpolator for a WSS input. Such a cyclostationary signal can be stationarized by an LTI system subject to certain conditions. This work aims to generalize this by characterizing an LTI system which can stationarize the output of a general fractional sampling filter.

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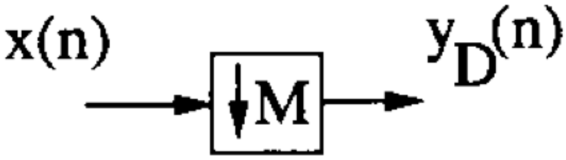
Chapter 1

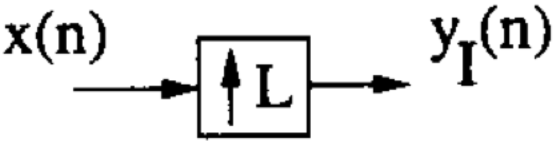
Introduction

Cyclostationary signals are stochastic processes. These processes are not periodic functions of time themselves rather they exhibit periodically varying statistical properties [1]. Such processes occur as output and intermediate signals in Multirate digital filtering which is used in a variety of applications such as subband coding, voice privacy systems, transmultiplexers, and adaptive filtering [2] to name a few. Generally in multirate digital signal processing, time-varying linear systems such as decimators, interpolators, and modulators are put together in complicated interconnections with linear filters which are time invariant [2].

Some work has been done previously to understand the way in which the statistical behavior of a signal changes as it passes through such systems. However this work addresses the problem that how one may convert a Cyclostationary process to a stationary process by passing it through an Linear Time Invariant system.



(a) M-fold Decimator



(b) L-fold Interpolator

FIGURE 1.1: Block Diagrams for Basic Multirate systems

1.1 Literature Survey

We now present a brief overview of the existing literature in the area of Study of Cyclostationary signals and multirate systems and specifically the effect of Multirate systems on the statistical properties of cyclostationary signals. A concise survey of the literature on cyclostationarity is presented in the work of Gardner et. al. [1]. A detailed analysis of pseudocirculants for LPTV system simplification, can be found in the Pioneering work of P. P. Vaidyanathan and Sanjit K. Mitra [3]. Other relevant contributions of Vaidyanathan with his Co-authors are as follows. His book *Multirate Systems and Filter Banks* [4] provides a good introduction to the field. His work with Vinay P. Sathe studies the effect of multirate systems on the second order statistical properties of stochastic signals [2]. In a later work done in 2000 with Sony he generalizes the above results by using the technique of Bispectra and Bifrequencies [5]. Two sources were referred to understand the concept of Bispectrum and Bifrequency maps. First was the 1984 paper by Loeffler and Burrus [6] and second was the book *Multirate Digital Signal Processing* by Crochiere [7]. Lastly the chapter *Cyclostationary Signal analysis* [8] in *Digital Signal Processing Handbook* was found to be very informative and helped clear up confusion in the different forms of Cyclostationary signal representation that may arise because the names used to refer to the same representations are often different in different works.

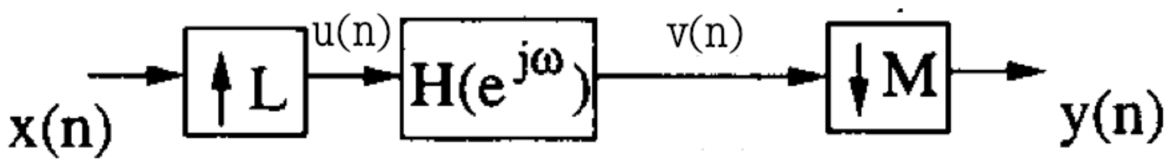


FIGURE 1.2: Fractional sampling filter

1.2 Outline

The report is organized as follows. In Chapter 2 we provide a review of the definitions, properties and results that we would use in our work. We cover the definitions of stationary and cyclostationary discrete random processes. We also include definitions of various multirate concepts such as Decimators Interpolators and Fractional Sampling filters. After that a review of polyphase decomposition of signals and systems is given. Then the effects of Blocking are studied on both signals and systems. Relevant results and derivations are given alongside the definitions for easy referencing. Lastly we review the Pseudocirculant conditions and the Stationarity condition on a Fractional Sampling filter such that it gives stationary output for a stationary input.

1.3 Approach and Contributions

We first define a term to conveniently denote the conditions on an lti system such that when it is cascaded to a system h then a stationary signal applied as input to the cascade produces a stationary output. Then we state our problem statement in terms of this term. We address the problem of specifying the above mentioned conditions for a Fractional sampling filter. In order to do that, analysis of the problem was done by looking at the bispectra of a general Fractional sampling filter. Several interesting conclusions were drawn from this study of the bispectra. The final solution was then found out by using the powerful Nobel identities and by using a previous result given by P.P. Vaidyanathan and Vinay Sathe in [2] for the simple Interpolator.

1.4 Notations used

Superscripts $(^*)$, $(^T)$ and $(^\dagger)$ denote the complex conjugate, matrix (or vector) transpose, and conjugate transpose respectively. Boldface letters are used for matrices and vectors. The (i, j) element of a matrix $\mathbf{B}$ is denoted by $[\mathbf{B}]_{(i,j)}$. Lower-case letters are used for 1-D and 2-D discrete sequences, whereas upper-case letters are used for 1-D and 2-D Fourier transforms. $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and real numbers respectively. The space of all finite norm M-component vector sequences is denoted by $l^2(M)$. [The l^2 norm of a vector sequence $\mathbf{x}(n)$ is defined as $||\mathbf{x}(n)|| = [\sum_n \mathbf{x}^\dagger(n)\mathbf{x}(n)]^1/2.$] Multirate system blocks are denoted by their common symbols and are illustrated in the Fig. 1.1(a), Fig. 1.1(b) and Fig. 1.2

Chapter 2

Preliminaries

We will review basic concepts and definitions and also build some results which would be used in later chapters of the report.

2.1 Introduction

Multirate signal processing is an active area of research having applications in communications, multimedia, etc., [4]. In our report we would be working with digital signals that have been sampled at different rates. Multiple rates due to the following two scenarios: (a) Altering the rate of a digital signal using traditional upsampling and downsampling, commonly termed as resampling, (b) Sampling an analog signal at different rates. Our work would be dealing with systems of the first category. Therefore we would now review the general multirate signal processing basics. Our work would also be focused on combining these results to manipulate cyclostationary signal , which we briefly mentioned in the beginning of the first chapter, into becoming stationary. Therefore in this chapter we would now set up all the definitions related to this area as well.

2.2 Wide sense stationary process

A vector stochastic process $\mathbf{x}(n)$ is said to be Wide sense stationary process if

1) $E[\mathbf{x}(n)] = E[\mathbf{x}(n+k)]$ for all integers n and k; and

2) the autocorrelation function depends only on the time difference between the two samples, i.e.,

$$E[\mathbf{x}(n)\mathbf{x}^{\dagger}(n-k)] = \mathbf{R}_{xx}(k), \forall n, \forall k. \tag{2.1}$$

The mean value $E[\mathbf{x}(n)]$ will usually not enter our discussion because it is normally assumed to be zero.

2.3 Cyclostationary process

Let $R_{xx}(n,k) = E[x(n)x^{*}(n-k)]$ denote the autocorrelation function of a scalar stochastic process x(n). The process is said to be $(CWSS)_L$ if

$$E[x(n)] = E[x(n+kL)], \forall n, \forall k \tag{2.2a}$$

$$R_{xx}(n,k) = R_{xx}(n+L,k), \forall n, \forall k. \tag{2.2b}$$

(We reiterate that mean values such as (2.2a) will not enter our discussions, as they are normally assumed to be zero.)

2.4 Power spectral density

The power spectral density $\mathbf{S}_{xx}(z)$ of a vector WSS process $\mathbf{x}(n)$ is defined as the z transform of its autocorrelation matrix defined in (2.1), i.e.,

$$\mathbf{S}_{xx}(z) = \sum_{k=-\infty}^{\infty} \mathbf{R}_{xx}(k) z^{-k} \tag{2.3}$$

Thus, each entry of this matrix is the z transform of the corresponding entry of $\mathbf{R}_{xx}(k)$. correspondingly $S_{xx}(e^{j\omega})$ is given by substituting z in $S_{xx}(z)$ by $(e^{j\omega})$.

2.5 General Linear Time Varying systems

A MIMO LTV system [5] with input $\mathbf{x}(n)$ and output $\mathbf{y}(n)$ is fully specified by the time-domain relation

$$\mathbf{y}(m) = \sum_{n=-\infty}^{\infty} \mathbf{k}(m,n)\mathbf{x}(n) = \sum_{n=-\infty}^{\infty} \mathbf{h}(m,n)\mathbf{x}(m-n) \tag{2.4}$$

Here, $\mathbf{k}(m,n)$ is called the *Green's function* and is perfectly general. $\mathbf{k}(m,n)$ specifies the response at time instance m the result of an impulse at time instance n. The function $\mathbf{h}(m,n)$ is the time-varying impulse response that is useful only if the input and output rates are equal. These are related as

$$\mathbf{h}(m,n) = \mathbf{k}(m, m-n) \tag{2.5}$$

2.5.1 LSIV systems

In multirate signal processing we encounter linear dual-rate systems, while altering the sampling rate. These systems are also called Linear Shift Invariant Systems (LSIV) systems. Basically in LSIV systems the *Green's function* is periodic in both the dimensions but with different periods. We call a system as $(LSIV)_{p,q}$ if the system's *Green's function* satisfies the following relation.

$$\mathbf{k}(m,n) = \mathbf{k}(m+p, n+q) \tag{2.6}$$

It should also be noted that by definition a period of (p, q) implies that if the input is shifted by time q then the output would be shifted by time p. This can be easily verified.

2.5.2 LPTV and LTI systems

LPTV systems are a sub class of LSIV systems (section 2.5.1) where p is equal to q. By replacing n by (m-n) in (2.6) and substituting (2.5) we get

$$\mathbf{h}(m,n) = \mathbf{k}(m+p, m+p-n) \tag{2.7a}$$

$$= \mathbf{h}(m+p, n) \tag{2.7b}$$

This implies that an LPTV system requires at most p infinite impulse responses to be fully characterized. The case of LTI systems is even more special since in their case p=q=1 and the *Green's function* **k** and impulse response **h** reduces to just a single function

$$\mathbf{k}(m,n) = \mathbf{h}(m-n) \tag{2.8}$$

2.6 Bifrequency maps

The LTV system described above is also fully specified by the bifrequency function which is simply the two dimensional DTFT of the *Green's function*

$$\mathbf{K}(e^{j\omega'}, e^{j\omega}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{k}(m,n) e^{-j\omega' m} e^{-j\omega n} \tag{2.9}$$

The system input-output relation in the frequency domain is

$$\mathbf{Y}(e^{j\omega'}) = \int_{-\pi}^{\pi} \mathbf{K}(e^{j\omega}, e^{j\omega}) \mathbf{X}(e^{j\omega}) d\omega \tag{2.10}$$

Cascading two LTV systems with Greens functions $\mathbf{k}_i(m,n)$ and bifrequencies $\mathbf{K}_i(e^{j\omega'}, e^{j\omega}), i = 1, 2$ (in that order) gives a new LTV system with Greens function $\mathbf{k}(m,n)$ and bifrequency $\mathbf{K}(e^{j\omega'}, e^{j\omega})$ given by

$$\mathbf{k}(m,n) = \sum_{r=-\infty}^{\infty} \mathbf{k_2}(m,r) \mathbf{k_1}(r,n) \tag{2.11}$$

$$\mathbf{K}(e^{j\omega'}, e^{j\omega}) = \int_{-\pi}^{\pi} \mathbf{K_2}(e^{j\omega'}, e^{j\omega''}) \mathbf{K_1}(e^{j\omega''}, e^{j\omega}) d\omega'' \tag{2.12}$$

2.7 M-fold decimator

A decimator shown in Fig.1.1(a) is a $(LSIV)_{1,M}$ device that takes an input sequence $x(n)$ and produces the output sequence

$$y_D(n) = x(nM). \tag{2.13}$$

This means that only those samples of x(n) that occur at sample locations equal to integer multiples of M are retained. In the transform domain, the

Fourier transforms are related as

$$Y_D(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j\omega/M}W_M^k) \tag{2.14}$$

where $W_M = e^{-2j\pi/M}$. Thus, in general, decimation causes aliasing
We now state three results related to the statistical property of M-fold decimator output and its bispectrum that we would be using later on

1. The decimator output for a WSS input is WSS.
2. The decimator output for $(CWSS)_L$ process is $(CWSS)_P$, P = L/gcd(L, M) [2].
3. The bifrequency map of the *Green's function* $k(m, n)$ of a decimator is [7].

$$K(e^{j\omega'}, e^{j\omega}) = \sum_{l=-\infty}^{\infty} \delta(\omega' - \omega M - 2\pi l) \tag{2.15}$$

This mapping is illustrated in Fig.2.1 for M = 3. The Principal values of the mapping are illustrated by the solid lines and it is clear that aliasing is occurring in this process

2.8 L-fold interpolator

The interpolator shown in Fig.1.1(b) is a $(LSIV)_{L,1}$ device that takes an input sequence $x(n)$ and produces an output sequence

$$y_l(n) = \begin{cases} x(n/L), \texttt{if n is an integer multiple of L} \\ 0, \texttt{otherwise} \end{cases} \tag{2.16}$$

In the frequency domain, we can write

$$Y_l(e^{j\omega}) = X(e^{j\omega L}). \tag{2.17}$$

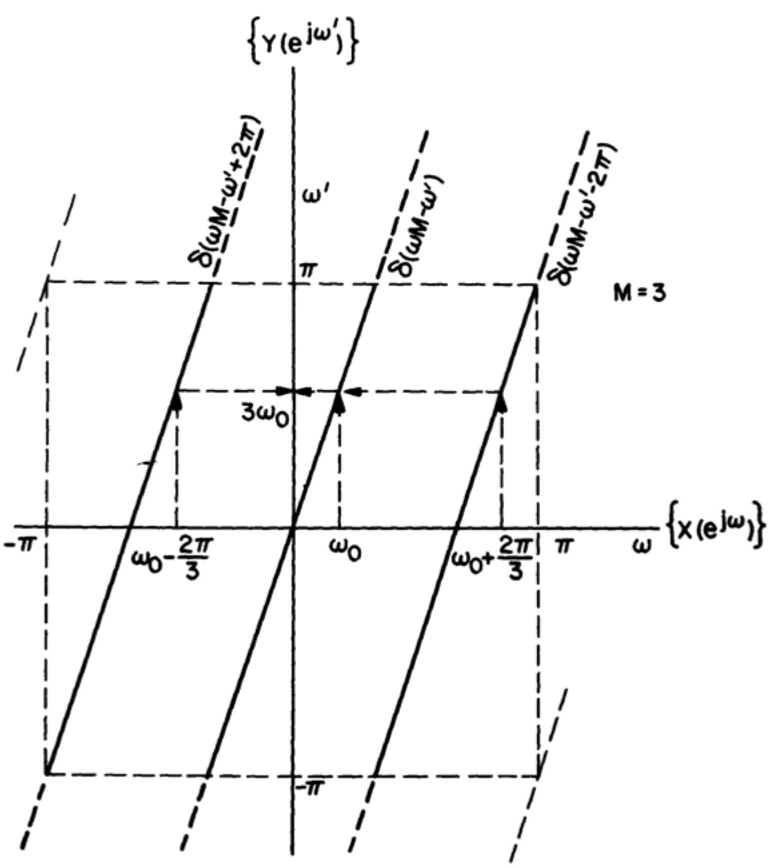


FIGURE 2.1: Bispectra of 3-fold Decimator

This means that we now have L squeezed copies of the spectrum of $X(e^{j\omega})$ in the region $0 \leq \omega < 2\pi$.

We now state three results related to the statistical property of L-fold interpolator output and its bispectrum that we would be using later on

1. The interpolator output for a WSS input is $(CWSS)_L$
2. The correlation sequence of the output $R_{yy}(\tau) = R_{xx}(\tau/L)$.

3. The bifrequency map of the *Green's function* $k(m,n)$ of an interpolator is [7]

$$K(e^{j\omega'}, e^{j\omega}) = \sum_{l=-\infty}^{\infty} \delta(\omega' - \omega/L - 2\pi/Ll) \tag{2.18}$$

This mapping is illustrated in Fig.2.2 for M = 3.

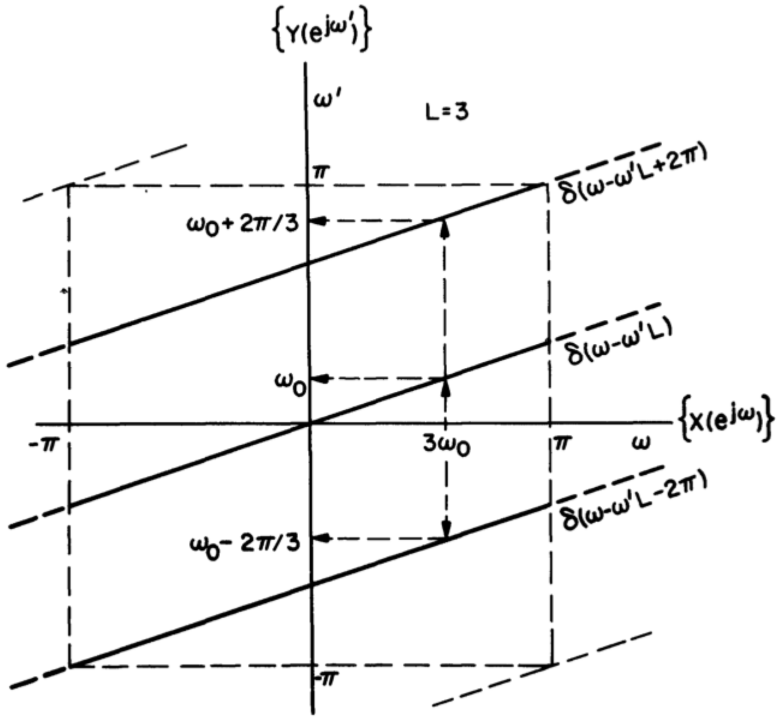


FIGURE 2.2: Bispectra of 3-fold Interpolator

2.9 Polyphase decomposition

The polyphase decomposition of an LTI transfer function $H(z)$ can be written in two forms. It is given by

$$H(z) = \sum_{i=0}^{M-1} z^{-i} E_i(z^M) \tag{2.19a}$$

$$= \sum_{i=0}^{M-1} z^{M-1-i} R_i(z^M) \tag{2.19b}$$

It may be noted that the two function E and R are related in the following manner

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots \tag{2.20a}$$

$$E_i(z) = R_{M-1-i}(z) = h_i + h_{M+i} z^{-1} + h_{2M+i} z^{-2} + \cdots \tag{2.20b}$$

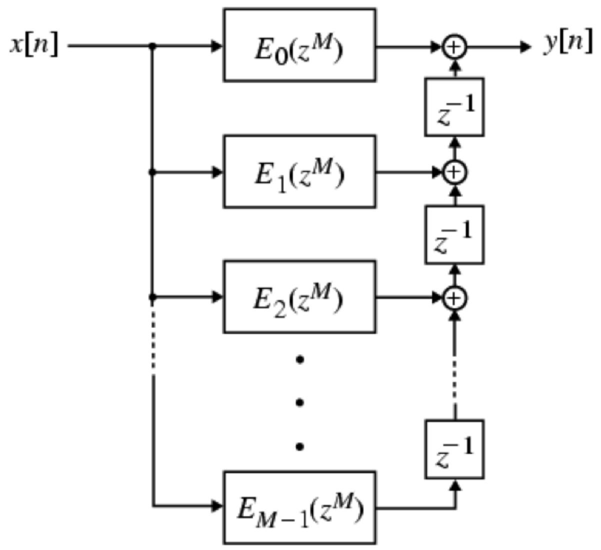
$E_i(z)$ is called the Type-I polyphase decomposition Fig.2.3(a), where $R_I(z)$ is called the type-II polyphase decomposition Fig.2.3(a).

The polyphase decomposition of any signal can be defined in an analogous manner. Let $X(z)$ be the z transform of a signal $x(n)$. The polyphase decomposition of $x(n)$ with respect to an integer M is

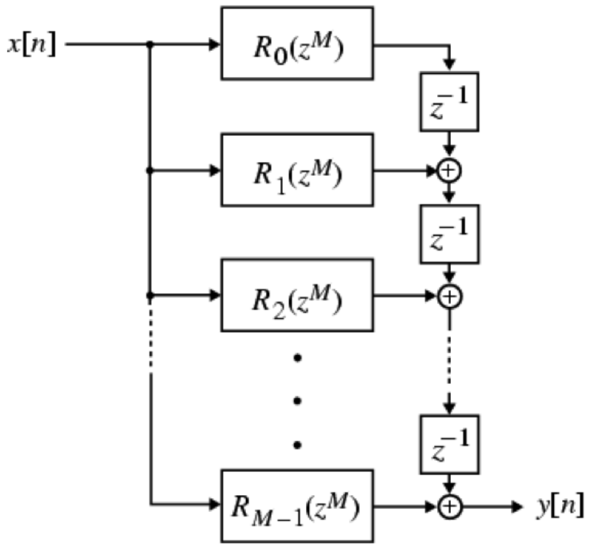
$$X(z) = z^{-(M-I)} R_0(z^M) + z^{-(M-2)} R_1(z^M) + \cdots + R_{M-1}(z^M). \tag{2.21}$$

Each function $R_i(z), 0 \leq i \leq M-1$ is called a polyphase component of $X(z)$. In the time domain, the kth polyphase component is obtained as

$$r_k(n) = x(nM + M - 1 - k). \tag{2.22}$$



(a) Type - I Polyphase decomposition



(b) Type - 2 Polyphase decomposition

2.10 Blocking

Blocking is a lossless operation, which associated a vector signal $\mathbf{x}(n)$ to a scalar signal $x(n)$, If we define the M-fold blocked version $\mathbf{x}(n)$ by

$$\mathbf{x}(n) = [x(nM), x(nM-1), \ldots, x(nM-M+l)]^T. \qquad (2.23)$$

then we can describe blocking operation as an operator $\mathbf{B}_M$

$$\mathbf{x} = \mathbf{B}_M x \Leftrightarrow \mathbf{x}(n) = [x_0(n), x_1(n), \ldots, x_{M-1} + (n)]^T \tag{2.24}$$

where $x_i(n) = x(Mn - i)$. The Block diagram implementation of the blocking operation B_M and the unblocking operation B_M^{-1} is shown in Fig. 2.3.

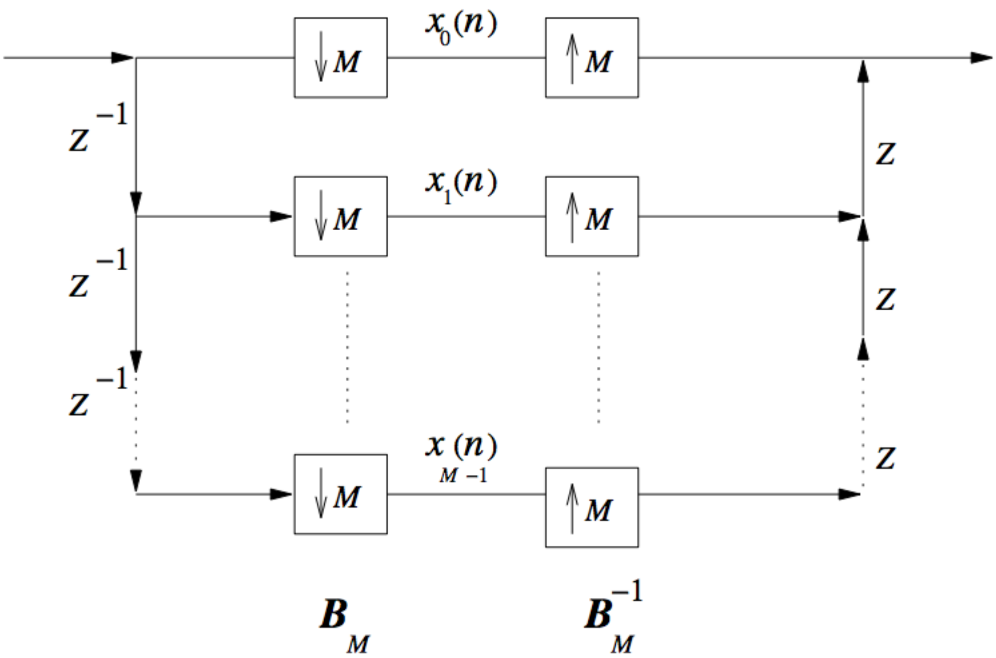


FIGURE 2.3: Multirate implementation of B_M and B_M^{-1}

2.11 Pseudocirculant matrices

An $M \times M$ matrix $\mathbf{A}(e^{j\omega})$ is said to be pseudocirculant if the entries $a_{i,l}(e^{j\omega})(i = 0, \ldots, M-1, l = 0, \ldots, M-1)$ satisfy the following relation:

$$a_{i,l}(e^{j\omega}) = \begin{cases} a_{0,l-i}(e^{j\omega}), 0 \leq i \leq l \\ e^{-j\omega} a_{0,l-i+M}(e^{j\omega}), l < i \leq M-1. \end{cases} \tag{2.25}$$

In words, a pseudocirculant matrix is a circulant, matrix with elements under the diagonal multiplied by $e^{j\omega}$. Here is an example of a 3×3 pseudocirculant matrix

$$\mathbf{A}(e^{j\omega}) = \begin{pmatrix} a & b & c \\ e^{-j\omega}c & a & b \\ e^{-j\omega}b & e^{-j\omega}c & a \end{pmatrix} \tag{2.26}$$

We will use the following properties of pseudocirculants, which can be verified from [3]:
1) If $\mathbf{A}(e^{j\omega})$ is pseudocirculant, then so is $\mathbf{A}^{\dagger}(e^{j\omega})$. If the inverse $[\mathbf{A}(e^{j\omega})]^{-1}$ exists, it is also pseudocirculant.
2) The product of pseudocirculant matrices is also pseudocirculant.

2.12 Stationarity condition for Fractional sampling filters

The Fractional sampling filter shown in Fig.1.2 is a $(LSIV)_{L,M}$ device made by a cascade of a L-fold interpolator, an LTI filter H and a M-fold decimator. We can clearly see that in general the output $y(n)$ would be $(CWSS)_{L/gcd(L,M)}$ for a WSS input $x(n)$ since the output of an LTI filter for a $(CWSS)_L$ process is also $(CWSS)_L$. This implies that if (Interpolator order) L = (Decimator order) M then the output would be $(CWSS)_1$ which is WSS, however if gcd(L, M) = 1 then the output would be $(CWSS)_L$

One question that was posed in relation to FS filters by P. P. vaidyanathan et. al. in [2] was "Can we design $H(e^{j\omega})$ such that the output $v(n)$ in Fig. 1.2 is WSS for a WSS input $x(n)$". The necessary and sufficient condition which characterizes such $H(e^{j\omega})$ is known as the "The Stationarity condition" and was the main result derived in [2]. "The Stationarity condition" states that the output $v(n)$ is WSS for a WSS input $x(n)$ if and only if no aliasing occurs if we perform L-fold decimation of the impulse response h(n). This condition is equivalent to the following condition: the frequency regions where $H(e^{j\omega})$ is nonzero do not overlap, if the frequency region $0 \leq \omega < 2\pi$ is reduced modulo $2\pi/L$. This condition automatically means that the output y(n) will be WSS too.

2.13 Concluding Remarks

Once we have the Stationarity condition for Fractional sampling filter at our hand, we are interested in the design of an appropriate lti filter that can be cascaded to stationarize the output of such a fractional sampling filter for a stationary input. If the fractional sampling filter satisfies the stationarity condition then this is trivial. The challenge lies in designing the system for the case when the fractional sampling filter does not satisfy the stationarity condition. One quest to do this led to the field of bispectral analysis where some interesting results were derived by us. The final solution was posed by us in terms of the well known stationarity condition of the simple interpolator.

Chapter 3

$\mathtt{C}LTI(h)$ condition for Fractional sampling filters

In this chapter we will take a new approach to a problem first considered by P. P. Vaidyanathan and Vinay P. Sathe in their paper "Effects of multirate systems on the statistical properties of random signals". They considered the problem that under what condition would the output of a Fractional sampling filter remain Stationary for a stationary input. The condition is called the "Stationarity Condition" which is mentioned in Section.2.12. This problem can be thought in a different way as finding a system which when cascaded to a Upsampler would always convert the output of the Upsampler to a stationary signal. In other words "Stationarize the Upsampler". In the following work we generalize that result by considering the problem of finding a system that can Stationarize the Fractional Sampling Filter (Shown in Fig. 1.2).

3.1 Introduction

The "Stationarity condition" mentioned in Section.2.12 gives us a condition on the LTI filter $h(n)$ such that the output $v(n)$ is stationary for stationary $x(n)$. Since the autocorrelation sequence of the input $u(n)$ is of the form $R_{uu}(\tau) = R_{xx}(\tau/L)$ we can interpret this condition in a different way as the condition on an LTI filter such that a particular special type of cyclostationary signal input to it is stationarized.

We now define a new term $\mathtt{C}LTI(h)$.
CLTI(h) $\triangleq$ Condition on an lti system such that when it is cascaded to a system h then a stationary signal applied as input to the cascade produces a stationary output.

If we denote an L-factor interpolator by $\uparrow L$ then The "Stationarity condition" mentioned above simply becomes the $\mathtt{C}LTI(\uparrow L)$ specific to an Interpolator. We aim to solve the problem of finding the $\mathtt{C}LTI(fsf)$ for the general Fractional sampling filter shown in Fig. 3.1 such that when we apply an LTI system g to the Fractional sampling filter then it gives a stationary output for a stationary input irrespective of whether $H(e^{j\omega})$ satisfies $\mathtt{C}LTI(\uparrow L)$ or not .

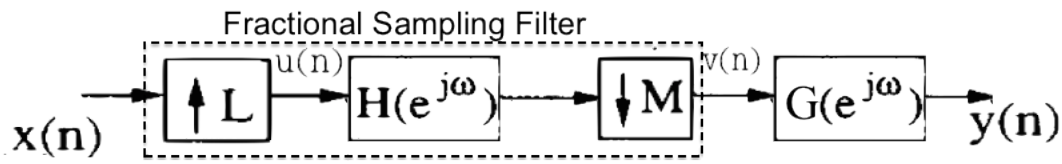


FIGURE 3.1: Our aim is to find $\mathtt{C}LTI$ condition for the system enclosed by dotted lines

3.2 Analysis

To analyze our problem we first look at the bispectra of a general Fractional sampling filter. Let $K_1(e^{j\omega'}, e^{j\omega})$, $K_2(e^{j\omega'}, e^{j\omega})$, $K_3(e^{j\omega'}, e^{j\omega})$ be the bifrequency maps of Interpolator, LTI filter and Decimator as they appear in the Fig. 1.2. The bispectra $K_2(e^{j\omega'}, e^{j\omega})$ can be simplified as [7]

$$K_2(e^{j\omega'}, e^{j\omega}) = H(e^{j\omega}) \sum_{l=-\infty}^{\infty} \delta(\omega' - \omega - 2\pi l) \tag{3.1}$$

The bispectra of the Fractional sampling filter is found by repeated application of (2.9). First we apply it to K_1 and K_2 to get K_t and substitute the value of K_1 from (2.18) and K_2 from above to get

$$K_t(e^{j\omega'}, e^{j\omega}) = \int_{-\pi}^{\pi} \left(H(e^{j\omega''}) \sum_{l_2=-\infty}^{\infty} \delta(\omega' - \omega'' - 2\pi l_2) \sum_{l_1=-\infty}^{\infty} \delta(\omega'' - \omega/L - 2\pi l_1/L) \right) d\omega'' \tag{3.2a}$$

$$= \sum_{l_2=-\infty}^{\infty} \sum_{l_1=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} H(e^{j\omega''}) \delta(\omega' - \omega'' - 2\pi l_2) \delta(\omega'' - \omega/L - 2\pi l_1/L) d\omega'' \right) \tag{3.2b}$$

Above equations imply that for fixed ω' and ω inner integral will be non-zero only for those pairs of l_1 and l_2 for which the following equations hold

$$\omega' - \omega'' - 2\pi l_2 = 0 \tag{3.3a}$$

$$\omega'' - \omega/L - 2\pi l_1/L = 0 \tag{3.3b}$$

for at least One value of ω'', where ω'' is constrained by

$$-\pi \leq \omega'' \leq \pi \tag{3.4}$$

Though a closed form solution of (3.2) is not possible for general $H(e^{j\omega''})$ we can draw following results.

1. Only those pairs of ω' and ω for which $(L\omega' - \omega)/2\pi = Ll_2 + l_1$ is possible would have a non-zero value. Since $Ll_2 + l_1$ can only take integer values this implies that the bispectrum would be non-zero only on those points for which $(L\omega' - \omega) = 2\pi k$. Also since the gcd of L and 1 is 1 therefore $Ll_2 + l_1$ can take any integer value which implies k can take any integer value.
2. Even though for each point (ω', ω) in the bispectrum we can generate infinite values of l_2 and l_1, the summation is constrained by (3.4) which would constrain the summation to be finite
3. Above points imply that the bispectrum of $K_t(e^{j\omega'}, e^{j\omega})$ is of the form $K_t'(e^{j\omega'}, e^{j\omega}) \sum_{l=-\infty}^{\infty} \delta(L\omega' - \omega - 2\pi l)$

Now we cascade K_t and K_3 substitute the value of K_3 from (2.15) to get

$$K(e^{j\omega'}, e^{j\omega}) = \int_{-\pi}^{\pi} \left(\sum_{l_3=-\infty}^{\infty} \delta(\omega' - \omega'' M - 2\pi l_3) K_t'(e^{j\omega''}, e^{j\omega}) \sum_{l_t=-\infty}^{\infty} \delta(L\omega'' - \omega - 2\pi l_t) \right) d\omega'' \tag{3.5a}$$

$$= \sum_{l_3=-\infty}^{\infty} \sum_{l_t=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} K_t'(e^{j\omega''}, e^{j\omega}) \delta(\omega' - \omega'' M - 2\pi l_3) \delta(L\omega'' - \omega - 2\pi l_t) d\omega'' \right) \tag{3.5b}$$

By doing a similar analysis as above we get the following results for the final bispectrum map

4. Only those pairs of ω' and ω for which $(L\omega' - M\omega)/2\pi = Ll_3 + Ml_t$ is possible would have a non-zero value. Since $Ll_3 + Ml_t$ can be expressed as $kgcd(L, M)$ this implies that the bispectrum would consist of parallel lines with slope M/L spaced apart by $2\pi gcd(L, M)/L$

5. Above points imply that the bispectrum of the complete system $K(e^{j\omega'}, e^{j\omega})$ is of the form
   $K'(e^{j\omega'}, e^{j\omega}) \sum_{l=-\infty}^{\infty} \delta(L\omega' - M\omega - 2\pi gcd(L, M)l)$

3.3 Solution

The system shown in Fig. 3.1 can be simplified by using the well known Polyphase identity by bringing filter $G(e^{j\omega})$ to the left hand side of the decimator. Fig. 3.2 shows the simplified block diagram of the entire system.

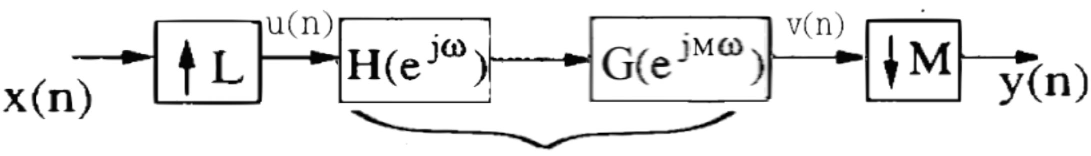


FIGURE 3.2: Simplification using well known Nobel identity

Now $H(e^{j\omega})$ and $G(e^{jM\omega})$ are multiplied since they are both LTI filters in cascade with each other. Also now we no longer have to focus on the decimator so we can discard the decimator to get the resulting block diagram shown in Fig. 3.3.

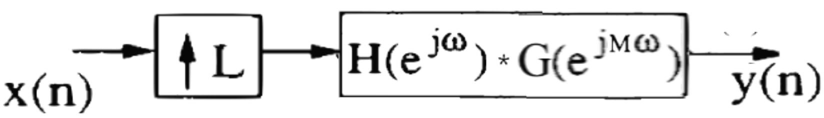


FIGURE 3.3: Final simplified Block diagram

Thus we have simplified the problem from finding $\mathtt{C}LTI(fsf)$ to finding the optimum $G(e^{j\omega})$ such that $H(e^{j\omega})G(e^{jM\omega})$ satisfies the $\mathtt{C}LTI(\uparrow L)$ condition. Now we analyze it further. Assume that $G(e^{j\omega})$ is Non-zero for only one frequency ω_0 in the interval $0 \leq \omega_0 < 2\pi$. Then its M-interpolated $G(e^{jM\omega})$ version would be non-zero at M such points in the interval $\{0, 2\pi\}$. The set of frequencies is $S : \{\omega_0/M, \omega_0/M+2\pi/M, \omega_0/M+4\pi/M, \ldots, \omega_0/M + 2(M-1)\pi/M\}$.

Now from $\mathbb{C}LTI(\uparrow L)$ discussed above we know that if the frequency regions where $G(e^{jM\omega})$ is non-zero do not overlap when the frequency region $0 \leq \omega_0 < 2\pi$ is reduced modulo $2\pi/L$. This implies that no two frequencies should overlap modulo $2\pi/L$ from set S defined above. We can easily see that for two frequencies from set S to overlap the following relation would have to hold

$$\frac{k_1}{M} = \frac{k_2}{L}, k_1 \in \{1, 2, \ldots, M-1\}, k_2 \in \mathbb{Z} \tag{3.6}$$

If M and L are co-prime then above equation (3.6) has no solution in the constraints however if M and L are not co-prime then it has a solution. Several results immediately follow

1. If L and M are co-prime then a narrow bandpass LTI filter would be able to stationarize the Fractional sampling filter. So at least one solution exists for our problem in this case.
2. If L and M are not co-prime then no LTI filter would be able to stationarize the Fractional sampling filter. So no solutions exist in this case.
3. This can be used as a test for whether the Interpolator and Decimator orders are co-prime or not in case we only know the ratio of input sample rate and output sample rate. Since there is no way to find out the gcd of the two even by using the bispectrum map of the fractional sampling filter.

3.4 Conclusions

The results shown above provide a way for designing optimum $G(e^{j\omega})$ such that the complete system consisting of the cascade of Fractional sampling filter and an LTI filter is stationarized. Of course, the flexibility enjoyed by the designer would depend upon $H(e^{j\omega})$ as to how far it deviates from the $\mathbb{C}LTI(\uparrow L)$ conditions and upon other design criteria. However, we have have also given an important result that such a design is simply not possible if M and L are not co-prime. The usage of these results would be illustrated further in our next chapter

Chapter 4

Concluding Remarks

In this report we have addressed the problem of Stationarizing general Fractional sampling filters and shown that it can only be done if the Interpolator order and Decimator order are co-prime. This implies that if the Interpolator and Decimator orders are not co-prime then no matter what LTI filter we put in cascade at the output of the Fractional sampling filter there would exist WSS inputs for which the output would be CWSS. We have also found the condition on an LTI filter such that it may stationarize a general Fractional sampling filter when cascaded to it. Subsequently We have also shown that this can be used as a test of whether the Interpolator and Decimator orders are co-prime or not. We have also studied and analyzed related problem in this work and presented the related theory in a structured manner with proofs as part of the Preliminaries chapter.

4.1 Scope for future research

The work presented in this report can be extended by generalizing the presented result for general fractional sampling filters to general Multirate

Filter bank. Since the sum of WSS processes need not be WSS itself therefore a simple extension of the above work would not be possible for general filter banks and other methods and techniques might have to be explored.

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